1. Taylor polynomials

1. Taylor polynomials
   1. The Taylor polynomial
   2. Error in Taylor’s polynomial
   3. Polynomial evaluation
Let $f(x)$ be a given function, for example

\[ e^x, \sin x, \log(x). \]

The Taylor polynomial mimics the behavior of $f(x)$ near $x = a$:

\[ T(x) \approx f(x), \text{ for all } x \text{ "close" to } a. \]

**Example**

Find a linear polynomial $p_1(x)$ for which

\[
\begin{align*}
    p_1(a) &= f(a), \\
    p_1'(a) &= f'(a).
\end{align*}
\]

$p_1$ is uniquely given by

\[ p_1(x) = f(a) + (x - a)f'(a). \]

The graph of $y = p_1(x)$ is tangent to that of $y = f(x)$ at $x = a$. 
Example

Let $f(x) = e^x$, $a = 0$. Then

$$p_1(x) = 1 + x.$$
Example

Find a **quadratic** polynomial $p_2(x)$ to approximate $f(x)$ near $x = a$.

Since

$$p_2(x) = b_0 + b_1 x + b_2 x^2$$

we impose three conditions on $p_2(x)$ to determine the coefficients. To better mimic $f(x)$ at $x = a$ we require

$$\begin{cases} 
  p_2(a) = f(a) \\
  p_2'(a) = f'(a) \\
  p_2''(a) = f''(a)
\end{cases}$$

$p_2$ is uniquely given by

$$p_2(x) = f(a) + (x - a)f'(a) + \frac{1}{2}(x - a)^2 f''(a).$$
Example

Let \( f(x) = e^x, a = 0 \). Then

\[
p_2(x) = 1 + x + \frac{1}{2}x^2.
\]

Figure: Linear and quadratic Taylor Approximation \( e^x \) (see eval_exp_simple.m)
Let \( p_n(x) \) be a polynomial of degree \( n \) that mimics the behavior of \( f(x) \) at \( x = a \). We require
\[
p_n^{(j)}(a) = f^{(j)}(a), \quad j = 0, 1, \ldots, n
\]
where \( f^{(j)} \) the \( j^{th} \) derivative of \( f(x) \).

The Taylor polynomial of degree \( n \) for the function \( f(x) \) at point \( a \):

\[
p_n(x) = f(a) + (x - a)f'(a) + \frac{(x - a)^2}{2!}f''(a) + \ldots + \frac{(x - a)^n}{n!}f^{(n)}(a)
\]

\[
= \sum_{j=0}^{n} \frac{(x - a)^j}{j!} f^{(j)}(a).
\]  

(3.1)

Recall the notations: \( f^{(0)}(a) = f(a) \) and the "factorial":
\[
j! = \begin{cases} 
1, & j = 0 \\
 j \cdot (j - 1) \cdots 2 \cdot 1, & j = 1, 2, 3, 4, \ldots
\end{cases}
\]
1. Taylor polynomials

1.1 The Taylor polynomial

Example

Let \( f(x) = e^x, \ a = 0. \)

Since \( f^{(j)}(x) = e^x, f^{(j)}(0) = 1, \) for all \( j \geq 0, \) then

\[
p_n(x) = 1 + x + \frac{1}{2!}x^2 + \ldots + \frac{1}{n!}x^n = \sum_{j=0}^{n} \frac{x^j}{j!} \tag{3.2}
\]

- For a fixed \( x, \) the accuracy improves as the degree \( n \) increases.
- For a fixed degree \( n, \) the accuracy improves as \( x \) gets close to \( a = 0. \)
Table. Taylor approximations to $e^x$

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<th>$x$</th>
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Example

Let $f(x) = e^x$, $a$ arbitrary, not necessarily 0. Since $f^{(j)}(x) = e^x$, $f^{(j)}(a) = e^a$, for all $j \geq 0$, then

$$p_n(x; a) = e^a \left(1 + (x - a) + \frac{1}{2!}(x - a)^2 + \ldots + \frac{1}{n!}(x - a)^n\right)$$

$$= e^a \sum_{j=0}^{n} \frac{(x - a)^j}{j!}$$
Example

Let \( f(x) = \ln(x) \), \( a = 1 \).

Since \( f(1) = \ln(1) = 0 \), and

\[
f^{(j)}(x) = (-1)^{j-1}(j-1)! \frac{1}{x^j}
\]

\[
f^{(j)}(1) = (-1)^{j-1}(j-1)!
\]

then the Taylor polynomial is

\[
p_n(x) = (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 - \ldots + (-1)^{(n-1)} \frac{1}{n} (x - 1)^n
\]

\[
= \sum_{j=1}^{n} \frac{(-1)^{j-1}}{j} (x - 1)^j
\]
1. Taylor polynomials

1.1 The Taylor polynomial

Figure: Taylor approximations of $\ln(x)$ about $x = 1$ (see plot_log.m)
Theorem (Lagrange’s form)

Assume \( f \in C^n[\alpha, \beta], x \in [\alpha, \beta] \).

The remainder \( R_n(x) \equiv f(x) - p_n(x) \), or error in approximating \( f(x) \) by \( p_n(x) = \sum_{j=0}^{n} \frac{(x - a)^j}{j!} f^{(j)}(a) \) satisfies

\[
R_n(x) = \frac{(x - a)^{n+1}}{(n + 1)!} f^{(n+1)}(c_x), \quad \alpha \leq x \leq \beta \tag{3.3}
\]

where \( c_x \) is an unknown point between \( a \) and \( x \).

[Exercise.] Derive the formal Taylor series for \( f(x) = \ln(1 + x) \) at \( a = 0 \), and determine the range of positive \( x \) for which the series represents the function.

Hint: \( f^{(k)}(x) = (-1)^{k-1} (k-1)! \frac{1}{(1+x)^k} \), \( f^{(k)}(0) = (-1)^{k-1} (k-1)! \), \( 0 \leq \frac{x}{1+c_x} \leq 1 \), if \( x \in [0, 1] \).

\[
\ln(1 + x) = \sum_{k=1}^{n} (-1)^{k-1} \frac{x^k}{k} + \frac{(-1)^{n}}{n+1} \frac{x^{n+1}}{(1+c_x)^{n+1}}
\]
Example

Let $f(x) = e^x$ and $a = 0$.

Recall that $p_n(x) = \sum_{j=0}^{n} \frac{x^j}{j!} = 1 + x + \frac{1}{2!}x^2 + \ldots + \frac{1}{n!}x^n$.

The approximation error is

$$e^x - p_n(x) = \frac{x^{n+1}}{(n+1)!}e^c, \quad n \geq 0 \tag{3.4}$$

with $c \in (0, x)$.

- **Exercise.** It can be proved that for each fixed $x$,

$$\lim_{n \to \infty} R_n(x) = 0, \text{ i.e., } e^x = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

(See the case $|x| \leq 1$).

- For each fixed $n$, $R_n$ becomes larger as $x$ moves away from 0.
Figure: Error in Taylor polynomial approx. to $e^x$ (see plot_exp_simple.m)
Example

Let \( x = 1 \) in (3.4), so from (3.2):

\[
e \approx p_n(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}
\]

and from (3.4)

\[
R_n(1) = e - p_n(1) = \frac{e^c}{(n+1)!}, \quad 0 < c < 1
\]

Since \( e < 3 \) and \( e^0 \leq e^c \leq e^1 \) we have

\[
\frac{1}{(n+1)!} \leq R_n(1) \leq \frac{e}{(n+1)!} < \frac{3}{(n+1)!}
\]
Suppose we want to approximate $e$ by $p_n(1)$ with

$$R_n(1) \leq 10^{-9}.$$ 

We have to take $n \geq 12$ to guarantee

$$\frac{3}{(n + 1)!} \leq 10^{-9},$$

i.e., to have $p_{12}$ as a sufficiently good approximation to $e$. 
Exercise

Expand $\sqrt{1 + h}$ in powers of $h$, then compute $\sqrt{1.00001}$.

If $f(x) = x^{\frac{1}{2}}$, then $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$, $f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$, $f'''(x) = \frac{3}{8}x^{-\frac{5}{2}}$, ...

$$\sqrt{1 + h} = 1 + \frac{1}{2}h - \frac{1}{8}h^2 + \frac{1}{16}h^3\epsilon^{-\frac{5}{2}}, \quad 1 < \epsilon < 1 + h, \text{ if } h > 0.$$ 

Let $h = 10^{-5}$. Then

$$\sqrt{1.00001} \approx 1 + .5 \times 10^{-5} - 0.125 \times 10^{-10} = 1.00000 49999 87500$$

Since $1 < \epsilon < 1 + h$, the absolute error does not exceed

$$\frac{1}{16}h^3\epsilon^{-\frac{5}{2}} < \frac{1}{16}10^{-\frac{5}{2}} = 0.00000 00000 00000 0625$$

and the numerical value is correct to all 15 decimal places shown.
Approximations and remainder formulae

\[ e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} e^c \quad (3.5) \]

\[ \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + (-1)^n \frac{x^{2n+1}}{(2n+1)!} \cos(c) \quad (3.6) \]

\[ \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + (-1)^{n+1} \frac{x^{2n+2}}{(2n+2)!} \cos(c) \quad (3.7) \]

\[ \frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \frac{x^{n+1}}{1-x}, \quad x \neq 1 \quad (3.8) \]

\[ (1 + x)^\alpha = 1 + \binom{\alpha}{1} x + \binom{\alpha}{2} x^2 + \cdots + \binom{\alpha}{n} x^n + \binom{\alpha}{n+1} x^{n+1} (1 + c)^{\alpha-n-1}, \quad \alpha \in \mathbb{R} \quad (3.9) \]

Recall that \( c \) is between 0 and \( x \), and the binomial coefficients are

\[ \binom{\alpha}{\kappa} = \frac{\alpha(\alpha - 1) \cdots (\alpha - \kappa + 1)}{\kappa!} \]
Example

Approximate $\cos(x)$ for $|x| \leq \frac{\pi}{4}$ with an error no greater than $10^{-5}$.

Since $\cos(c) \leq 1$ and

$$R_{2n+1}(x) \leq \frac{x^{2n+2}}{(2n + 2)!} \leq 10^{-5}$$

we must have

$$\frac{\left(\frac{\pi}{4}\right)^{2n+2}}{(2n + 2)!} \leq 10^{-5}$$

which is satisfied when $n \geq 3$. Hence

$$\cos(x) \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$
Indirect construction of a Taylor polynomial approximation

Remark

From (3.5), by replacing $x$ with $-t^2$, we obtain

\[
e^{-t^2} = 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \cdots + \frac{(-1)^n x^{2n}}{n!} + \frac{(-1)^{(n+1)} x^{2n+2}}{(n + 1)!} e^c,
\]

\[
- t^2 \leq c \leq 0.
\]

If we attempt to construct the Taylor approximation directly, the derivatives of $e^{-t^2}$ become quickly too complicated.
Indirect construction of a Taylor polynomial approximation

From (3.8) we can easily get:

\[-\int_0^t \frac{dx}{1-x} = -\int_0^t (1 + x + x^2 + \cdots + x^n) \, dx - \int_0^t \frac{x^{n+1}}{1-x} \, dx,\]

\[\ln(1-t) \overset{(3.12)}{=} - \left( t + \frac{1}{2} t^2 + \cdots + \frac{1}{n+1} t^{n+1} \right) - \frac{1}{1-c} \int_0^t x^{n+1} \, dx\]

\[\ln(1-t) = - \left( t + \frac{1}{2} t^2 + \cdots + \frac{1}{n+1} t^{n+1} \right) - \frac{1}{1-c} \frac{t^{n+2}}{n+2}, \quad (3.11)\]

where \(c_t\) is a number between 0 and \(t\), \(-1 \leq t \leq 1\).

**Theorem (Integral Mean Value Theorem)**

Let \(w(x)\) be a nonnegative integrable function on \((a, b)\), and \(f \in C[a, b]\). Then \(\exists\) at least one point \(c \in [a, b]\) for which

\[\int_a^b f(x)w(x) \, dx = f(c) \int_a^b w(x) \, dx \quad (3.12)\]
By rearranging the terms in (3.8) we obtain the sum of a infinite geometric series

\[ 1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}, \quad x \neq 1. \] (3.13)

For \(|x| < 1\), letting \(n \to \infty\) we obtain the infinite geometric series

\[ \frac{1}{1 - x} = 1 + x + x^2 + x^3 + \cdots = \sum_{j=0}^{\infty} x^j, \quad x \neq 1. \] (3.14)
Infinite Series

Definition

The infinite series

\[ \sum_{j=0}^{\infty} c_j \]

is **convergent** if the partial sums

\[ S_n = \sum_{j=0}^{n} c_j, \quad n \geq 0 \]

form a convergent sequence, i.e.,

\[ \exists S = \lim_{n \to \infty} S_n \]

and we then write

\[ S = \sum_{j=0}^{\infty} c_j \]
For the infinite series (3.14) with $|x| \neq 1$, the partial sums are given by (3.13):

$$S_n = \frac{1 - x^{n+1}}{1 - x}$$

- $S_n \xrightarrow{|x|<1} \lim_{n \to \infty} \frac{1}{1-x}$
- $S_n$ is divergent when $|x| > 1$
- What happens when $|x| = 1$?
Infinite Series

Definition

Assume that \( f(x) \) has derivatives of any order at \( x = a \). The infinite series

\[
\sum_{j=0}^{\infty} \frac{(x - a)^j}{j!} f^{(j)}(a)
\]

is called the **Taylor series expansion** of the function \( f(x) \) about the point \( x = a \).

The partial sum

\[
\sum_{j=0}^{n} \frac{(x - a)^j}{j!} f^{(j)}(a)
\]

is simply the Taylor polynomial \( p_n(x) \).
If the sequence $\{p_n(x)\}$ has the limit $f(x)$, i.e. the error tends to zero as $n \to \infty$

$$\lim_{n \to \infty} \left( f(x) - p_n(x) \right) = 0$$

then we can write

$$f(x) = \sum_{j=0}^{\infty} \frac{(x - a)^j}{j!} f^{(j)}(a)$$
Actually, it can be shown that the errors terms in (3.5)-(3.9) and (3.11) tend to 0 as \( n \to \infty \) for suitable values of \( x \). Hence the Taylor expansions

\[
e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}, \quad -\infty < x < \infty
\]

\[
\sin x = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+1}}{(2j + 1)!}, \quad -\infty < x < \infty
\]

\[
\cos x = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j}}{(2j)!}, \quad -\infty < x < \infty
\]

\[
(1 + x)^\alpha = \sum_{j=0}^{\infty} \binom{\alpha}{j} x^j, \quad -1 < x < 1
\]

\[
\ln(1 - t) = - \sum_{j=0}^{\infty} \frac{t^j}{j}, \quad -1 \leq x < 1
\]
Infinite series

Definition

Infinite series of the form
\[ \sum_{j=0}^{\infty} a_j (x - a)^j \] (3.16)

are called **power series**.

They can arise from Taylor’s formulae or some other ways. Their convergence can be examined directly.

Theorem (Comparison criterion)

Assume the series (3.16) converges for some value \( x_0 \). Then the series (3.16) converges for all \( x \) satisfying \( |x - a| \leq |x_0 - a| \).

Theorem (Quotient criterion)

For the series (3.16), assume that the limit
\[ R = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \]
exists. Then for \( x \) satisfying \( |x - a| < \frac{1}{R} \), the series (3.16) converges to a limit \( S(x) \). When \( R = 0 \), the series (3.16) converges for any \( x \in \mathbb{R} \).
Example

Consider the power series in (3.15). Letting $t = x^2$, we obtain the series

$$
\sum_{j=0}^{\infty} \frac{(-1)^j t^j}{(2j)!}
$$

(3.17)

Applying the quotient criterion with

$$
a_j = \frac{(-1)^j}{(2j)!}
$$

we find $R = 0$, so the series (3.17) converges for any value of $t$, hence the series in the formula (3.15) converges for any value of $x$. 

Infinite series
Consider the evaluation of the polynomial

\[ p(x) = 3 - 4x - 5x^2 - 6x^3 + 7x^4 - 8x^5 \]

**1** simplest method: compute each term independently, i.e.,

\[ c \times x^k \text{ or } c \times x \star \star k \]

yielding

\[ 1 + 2 + 3 + 4 + 5 = 15 \text{ multiplications} \]

**2** a more efficient method: compute each power of \( x \) using the preceding one:

\[ x^3 = x(x^2), \quad x^4 = x(x^3), \quad x^5 = x(x^4) \quad (3.18) \]

Since each term takes **two** multiplications for \( k > 1 \), the result will be

\[ 1 + 2 + 2 + 2 + 2 = 9 \text{ multiplications} \]

**3** nested multiplication:

\[ p(x) = 3 + x(-4 + x(-5 + x(-6 + x(7 - 8x)))) \]

with only **5** multiplications
Consider now the general polynomial of degree $n$:

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n, \quad a_n \neq 0$$

- If we use the $2^{nd}$ method, with the powers of $x$ computed as in (3.18), the number of multiplications in evaluating $p(x)$ is $2n - 1$.
- For the nested multiplication, we write and evaluate $p(x)$ in the form

$$p(x) = a_0 + x(a_1 + x(a_2 + \cdots + x(a_{n-1} + a_n x)\cdots)) \quad (3.19)$$

using only $n$ multiplications, saving about 50% over the $2^{nd}$ method.

All methods use $n$ additions.
Example

Evaluate the Taylor polynomial $p_5(x)$ for $\ln(x)$ about $a = 1$.

A general formula is (3.11), with $t$ replaced by $-(x - 1)$, yielding

$$p_5(x) = (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 - \frac{1}{4}(x - 1)^4 + \frac{1}{5}(x - 1)^5$$

$$= w \left( 1 + w \left( -\frac{1}{2} + w \left( \frac{1}{3} + w \left( -\frac{1}{4} + \frac{1}{5}w \right) \right) \right) \right).$$
A more formal algorithm than (3.19)

Suppose we want to evaluate \( p(x) \) at number \( z \). Let define the sequence of coefficients \( b_i \) as:

\[
\begin{align*}
    b_n &= a_n \quad \text{(3.20)} \\
    b_{n-1} &= a_{n-1} + zb_n \quad \text{(3.21)} \\
    b_{n-2} &= a_{n-2} + zb_{n-1} \quad \text{(3.22)} \\
    &\vdots \quad \text{(3.23)} \\
    b_0 &= a_0 + zb_1 \quad \text{(3.24)}
\end{align*}
\]

Now the nested multiplication is the **Horner Method**:

\[
p(z) = a_0 + z(a_1 + z(a_2 + \cdots + z(a_{n-1} + a_n z) \cdots))
\]
Hence

\[ p(z) = b_0 \]

With the coefficients from (3.20), define the polynomial

\[ q(x) = b_1 + b_2x + b_3x^2 + \cdots + b_{n-1}x^{n-1} \]

It can be shown that

\[ p(x) = b_0 + (x - z)q(x), \]

i.e., \( q(x) \) is the quotient from dividing \( p(x) \) by \( x - z \), and \( b_0 \) is the remainder.

**Remark**

*This property we will use it later for polynomial rootfinding method to reduce the degree of a polynomial when a root \( z \) has been found, since then \( b_0 = 0 \) and \( p(x) = (x - z)q(x) \).*