3.4.2 \[ \dot{x} = rx - \sinh x \]

\[ x = 0 \text{ is a fixed point for all } r \]

\[
\frac{d}{dx} \sinh x \bigg|_{x=0} = \cosh 0 = 1 \Rightarrow r_c = 1
\]

\[ r < 1 \quad \rightarrow \quad r > 1 \]

Supercritical pitchfork

3.4.9 \[ \dot{x} = x + \tanh(rx) \]

\[ \tanh(rx) = -x \quad x = 0 \text{ is a fixed point for all } x \]

\[
\frac{d}{dx} \tanh(rx) = \frac{d}{dx}(-x)
\]

\[ \frac{r}{\cosh(rx)} = -1 \quad \text{at } x = 0, \quad r_c = -1
\]

\[ r < r_c \quad \rightarrow \quad r > r_c \]

Subcritical pitchfork
3.4.10 \[
\dot{x} = r x + \frac{x^3}{1 + x^2} = \left( r + \frac{x^2}{1 + x^2} \right) x
\]

Fixed points: 
\[ x = 0 \]
\[ r(1 + x^2) = -x^2 \]
\[ x^2 = -1 - r \]
\[ x_{\pm} = \pm \sqrt{1 - r} \]

Subcritical pitchfork for \(-1 < r \leq 0\)

bifurcation diagram

3.4.14 \[
\dot{x} = r x + x^3 - x^5
\]

a) Fixed points: 
\[ x = 0 \]
\[ r + x^2 - x^4 = 0 \Rightarrow x^2 = \frac{1 \pm \sqrt{1 + 4r}}{2} \]

Real for \(r \geq \frac{1}{4}\)

4 solutions for \(-\frac{1}{4} < r < 0\)

Saddle node at \(r_c = -\frac{1}{4}\)

Subcritical pitchfork at \(r_c = 0\)

bifurcation diagram
\[ u = au + bu^3 - cu^5 \quad b, c > 0 \]

Let \( u = Vx \), \( \dot{T} = T\dot{\theta} \)

\[ \dot{u} = \frac{V}{T} \frac{dx}{d\tau} = aVx + bV^3x^2 - cV^5x^5 \]

\[ \text{divide by} \ V \]

\[ V = \sqrt{\frac{b}{c}} \]

Let \( V \) be such that \( b = cV^2 \)

Then
\[ \frac{1}{\sqrt{\frac{b}{c}}} \frac{dx}{d\tau} = aVx + b\left(\frac{b}{c}\right)^{\frac{3}{2}}(x^3 - x^5) \]

Divide by \( \sqrt{\frac{b}{c}} \)

Let
\[ \frac{1}{T} \frac{dx}{d\tau} = aVx + \frac{b^2}{c}(x^3 - x^5) \]

Let \( T = \frac{c}{b^2} \)

\[ \frac{dx}{d\tau} = \frac{ac}{b^2}x + x^3 - x^5 \]

Let \( r = \frac{ac}{b^2} \)

\[ r\dot{x} + x^3 - x^5 \]
$x = rx + ax^2 - x^3$

Fixed points: $x = 0$

$r + ax - x^2 = 0 \Rightarrow x = \frac{a \pm \sqrt{a^2 + 4r}}{2}$ real if $r > -\frac{a^2}{4}$

a) Bifurcation diagrams

- $a < 0$
- $a = 0$
- $a > 0$

Transcritical at $r = 0$
Saddle node at $r = -\frac{a^2}{4}$

b)

3 fixed pts

r

Transcritical

$\rightarrow a$

3 fixed pts.

r = -\frac{a^2}{4}$

Saddle-node
\[ \dot{g} = \frac{g_1 s_0 - g_2 g + \frac{g_2 g^2}{g_1 + g^2}}{g_1 + g^2} \]

9) Let \( g = A \tau \), \( t = B \tau \)

\[ \dot{\tau} = \frac{A}{B} \frac{dx}{dt} = \frac{A}{B} (s_0 - A x^2) + \frac{A^2 x^2}{g_1 + A x^2} \]

Let \( A = g_4^2 \)

\[ \frac{g_4^2}{B} \frac{dx}{dt} = g_1 s_0 - g_2 g_4 x^2 + \frac{g_3 x^2}{1 + x^2} \]

Let \( B = \frac{g_4^2}{g_3} \)

\[ \frac{dx}{dt} = \frac{g_1 s_0}{g_3} - \frac{g_2 g_4 x^2}{g_3} + \frac{x^2}{1 + x^2} \]

Let \( s = \frac{g_1 s_0}{g_3} \), \( r = \frac{g_2 g_4}{g_3} \)

b) For \( s = 0 \)

\[ x = -r x + \frac{x^2}{1 + x^2} \]

Fixed points:

\[ x = 0 \]

\[ -r(1 + x^2) + x = 0 \]

\[ \alpha_{1,2} = \frac{-1 \pm \sqrt{1 - 4r^2}}{-2r} \]

2 fixed points for \( 0 < r < r_c = \frac{1}{2} \)

\[ \text{saddle-node bifurcation} \]

\[ r \]

\[ \frac{1}{2} \]
c) \[ x = S - rx + \frac{x^2}{1+x^2} \]

Assume \( S \) is the bifurcation parameter, \( r \) is fixed.

Bifurcation diagram

When \( g(0) = 0 \), solution is on the lower stable branch. As \( S \) is slowly increased, it moves towards the bifurcation point and then jumps onto the upper branch. It will remain there as \( S \) is decreased back to 0.

When \( g(0) = 0 \), solution is on the lower stable branch. As \( S \) is slowly increased, it moves towards the bifurcation point and then jumps onto the upper branch. It will remain there as \( S \) is decreased back to 0.

\[ 0 = \frac{d}{dx} S = \frac{d}{dx} \left( rx - \frac{x^2}{1+x^2} \right) = r - \frac{2x}{(1+x^2)^2} = 0 \]

\[ r = \frac{2x}{(1+x^2)^2} \]

Substitute back

\[ S = rx - \frac{x^2}{1+x^2} = \frac{2x^2}{(1+x^2)^2} - \frac{x^2}{1+x^2} = \frac{x^2(1-x^2)}{(1+x^2)^2} \]