1. The transition matrix for an infinite state DTMC is \( P = \begin{pmatrix} a & a & a & \cdots \\ 1-a & 0 & 0 & \cdots \\ 0 & 1-a & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \) \( 0 < a < 1 \)

(a) Show that the DTMC is irreducible and aperiodic.
(b) Show that the first two rows of \( P^2 \) are constant, i.e., \( p_{1,j}^{(2)} = a, p_{2,j}^{(2)} = a(1-a) \). Then show that the first three rows of \( P^4 \) are constant. What is \( p_{3,j}^{(4)} \)?
(c) Show that the Markov chain is recurrent and find the limit \( \lim_{n \to \infty} P^n \).
(d) Is the chain positive recurrent or null recurrent?

2. Consider a discrete time random walk on the finite set \( \{0,1,2,...,N\} \) with absorbing boundaries and with the following transition probabilities: \( p_{i+1,i} = p, p_{i-1,i} = q, p_{i,i} = r \), \( i = 1,...,N-1 \); \( p_{00} = p_{NN} = 1 \)

(a) Show that the probability \( a_k \) of absorption at \( x = 0 \) and the expected time to absorption \( \tau_k \), for a chain starting at \( k \), are
\[
 a_k = (q/p)^N - (q/p)^{k-1}, \quad \tau_k = \frac{1}{q-p} \left[ k - N \frac{(q/p)^{k-1}}{1-(q/p)^{N-1}} \right], \quad q \neq p
\]
(b) Find the values of \( a_k \) and \( \tau_k \) for the case when \( q = p \).

3. Consider a death and immigration CTMC process, where \( \lambda_i = \nu, \mu_i = \mu i \)

(a) Derive the PDE for the p.g.f. \( P(z,t) = \sum_{i=0}^{\infty} p_i(t) z^i \)
(b) Use the method of characteristics to solve the PDE with initial condition \( P(z,0) = z^N \)
(c) Find the mean and the variance of the process

4. Consider the CTMC birth process with immigration, where \( \lambda_n = b_0 + b_1 n^k \), \( b_0, b_1 > 0 \). Show that the process is not explosive if \( k = 1 \) and is explosive if \( k = 2,3,... \)

5. Consider the SDE for the Ornstein-Uhlenbeck process \( dX(t) = aX \ dt + b \ dW(t), \ \ X(0) = x_0 \)
where \( a, b, x_0 \) are constants and \( b > 0 \).

(a) Solve for \( X(t) \) using Itô formula. (Hint: Let \( F(X,t) = X \exp(-at) \).)
(b) What are the mean and variance of this process?

6. Consider diffusion process with drift on the interval \( (A,B) \), with backward Kolmogorov equation
\[
\frac{\partial p(y,t \mid x,0)}{\partial t} = c \frac{\partial p(y,t \mid x,0)}{\partial x} + \frac{D}{2} \frac{\partial^2 p(y,t \mid x,0)}{\partial x^2}
\]
(a) Find the probability \( Q(x) \) of reaching state \( A \) before state \( B \).
(b) Find the expected time \( m_T(x) \) of reaching \( A \) or \( B \).