If \( f(t) = e^t \) and \( g(t) = e^{-t} \), then \( \int_0^t f(u) g(t-u) \, du = \int_0^t e^{u} e^{-ut} \, du \).

\[
\frac{d}{dt} [e^{u} - e^{-ut}] = -e^{-ut} - (-e^t) = e^t - e^{-t}
\]

\[
\text{Integrating by parts: } \int_0^t f(u) g(t-u) \, du = \left[ -e^{-ut} \right]_0^t + \int_0^t e^{-ut} \, du = -te^{-t} - e^{-t} + t - 1 + e^t.
\]

5.7.10 a) If \( f(t) = e^t \) and \( g(t) = e^{-t} \), then \( F(1) = 1/(e-1) \) & \( G(1) = 1/(e^2) \). Therefore,

\[
F(1) G(1) = \left( \frac{1}{e-1} \right) \left( \frac{1}{e^2} \right) = \frac{1}{(e-1)(e^2)} = \frac{1}{2e^3 - 3e^2 + 1}
\]

b) \( \mathcal{L}^{-1} \{ F(t) G(t) \} = \mathcal{L}^{-1} \{ e^{2t} \} = \frac{1}{2} e^t \)

Both (a) & (b) emphasize the fact that the Laplace transform of the product of two functions is not the product of their Laplace transforms.

8.1.5 The system \( u_1' = u_2 \) & \( u_2' = -\frac{1}{2} u_2 + \frac{1}{4} u_3 \), \( u_3' = u_4 \), \( u_4' = \frac{3}{2} u_1 + \frac{1}{4} u_3 \)

has dependent variables (unknowns) \( u_1, u_2, u_3 \) and \( u_4 \). Therefore, the dimension is 4. The right side does not depend explicitly on \( t \), so the system is autonomous.

8.1.7 \( x' = (2e^{3t} - 2e^{-t}) \), \( y' = (-e^{-t} + 2e^{3t}) = e^{-t} + 4e^t \)

\[
\begin{align*}
4x + 6y &= -4(2e^{3t} - 2e^{-t}) + 6(-e^{-t} + 2e^{3t}) = 4e^{3t} + 2e^{-t} = x,

-3x + 5y &= -3(2e^{3t} - 2e^{-t}) + 5(-e^{-t} + 2e^{3t}) = 4e^{3t} + e^{-t} = y.
\end{align*}
\]

\[
x(0) = 2e^{3(0)} - 2e^{-0} = 0, \quad y(0) = -e^{-0} + 2e^{3(0)} = 1
\]

8.1.13 We know that \( x' = -y + x - x^2 + y \) is a system, \( x(0) = x_0, x'(0) = v_0 \) if we let \( u_2 = x, u_3 = y' \), then \( u_2' = u_3 & \; u_3' = -2u_2 + u_2 - u_2 u_3 + x' \) counts.

Furthermore, \( u_2(0) = x(0) = x_0 & \; u_3(0) = x'(0) = v_0 \).

8.1.15 We know that \( w'' = w, w(0) = w_0, w'(0) = x_0 \), and \( w''(0) = y_0 \). If we let \( u_1 = w, u_2 = w' \), and \( u_3 = w'' \), then \( u_2' = u_3 & \; u_3' = -u_2 + u_3 \) counts.

Furthermore, \( u_1(0) = w(0) = w_0, u_2(0) = w'(0) = x_0 \), and \( u_3(0) = w''(0) = y_0 \).

8.1.12 If \( x(0) = (e^{-t} \cos t, e^{-t} \sin t) \), then \( x'(0) = (e^{-t}(-\sin t - \cos t), e^{-t}(-\cos t + \sin t)) \).

In the figure that follows, the derivative was used to plot vectors tangent to the curve at selected points. Tangent vectors are plotted at 15% of their actual length.
\[ x' = y \]
\[ y' = -\sin(x) \]

5.7.6
\[ f(t) = t - 1, \quad g(t) = t - 2 \]
\[ f \ast g(t) = \int_0^t f(u) g(t-u) \, du \]
\[ = \int_0^t (u-1)(t-u-2) \, du \]
\[ = \int_0^t (u^2 - u^3 - u^2 - t^2 + u) \, du \]
\[ = \left. \frac{u^3}{3} - \frac{u^4}{4} - \frac{u^2}{2} - tu + 2u \right|_0^t \]
\[ = \frac{t^3}{3} - \frac{t^4}{4} - \frac{t^2}{2} - t^2 + 2t \]
\[ = \frac{t^3}{3} - \frac{t^4}{4} + 2t \]
\[ x' = (0.4 - 0.01y)x \]
\[ y' = (0.005x - 0.3)y \]