Problem 1: Construct the center manifold and discuss the bifurcation at $\varepsilon = 0$.

(a) $\dot{x} = y + x^2 + \varepsilon \alpha$
\hspace{1cm} $\dot{y} = -y - x^2$

(b) $\dot{x} = y + \varepsilon \alpha + x^3$
\hspace{1cm} $\dot{y} = -y + x^3$

(c) $\dot{x} = \varepsilon + \varepsilon \alpha + x^2$
\hspace{1cm} $\dot{y} = \varepsilon - y + x^2$

Problem 2: Consider the system $\dot{x} = f(x, \mu)$ with $(x, \mu) \in \mathbb{R}^2 \times \mathbb{R}$ with fixed point at $(0,0)$ and with $\nabla_x f(0,0)$ having one zero and one negative eigenvalue. Suppose the vector field has the following symmetry: $f(x, \mu) = -f(-x, \mu)$. What can you conclude about the center manifold and the type of bifurcation the system can undergo at $(0,0)$? Justify your claims.

Problem 3: Consider the system

\[ \dot{x} = -x^4 + 5 \mu x^2 - 4 \mu^2 \]

Determine the bifurcations, draw the bifurcation diagram and the phase portraits for various values of $\mu$.

Problem 4: Check that the following system has an equilibrium that exhibits the Hopf bifurcation and compute the first Lyapunov coefficient

\[ \dot{x} = y \]
\[ \dot{y} = -x + \alpha y + x^2 + xy + y^2 \]