Topics

Review of lecture 2/11
Error, Residual and Condition Number

Review of lecture 2/16
Backward Error Analysis
The General Case
Theorem (Calculation of 2 norm of a symmetric matrix)

If $A = A^t$ is symmetric then $\|A\|_2$ is given by

$$\|A\|_2 = \max\{|\lambda| : \lambda\ is\ an\ eigenvalue\ of\ A\}.$$ 

Theorem (Matrix 1-norm and $\infty$-norm)

We have

$$\|A\|_\infty = \max_{1 \leq i \leq N} \sum_{j=1}^{N} |a_{ij}|,$$

$$\|A\|_1 = \max_{1 \leq j \leq N} \sum_{i=1}^{N} |a_{ij}|.$$
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Approximate solution $\hat{x}$ of $Ax = b$

\[
\begin{align*}
\text{error} & : \quad e = x - \hat{x}, \\
\text{residual} & : \quad r = b - A\hat{x}.
\end{align*}
\]

Small residual need not imply small error

the point $P = (0.5, 0.7)$ and the $2 \times 2$ linear system

\[
\begin{align*}
x - y & = 0 \\
-0.8x + y & = 1/2.
\end{align*}
\]

Plot is two almost-parallel lines with $\hat{x}$ between them, far from the solution.
Normalize each equation in the system so that $\sqrt{a_{11}^2 + a_{12}^2} = 1$

Normal distance from $\hat{x}$ to the line is $r_1 = b_1 - a_{11}\hat{x}_1 - a_{12}\hat{x}_2$

Hence $\|r\|_2 =$ square root of sum of squares of normal distance from $\hat{x}$ to lines.

Same for 3D and higher dimensions
Recall qualitative facts

- If $A$ is invertible then $r = 0$ if and only if $e = 0$.
- If $A$ is well conditioned then $\|r\|$ and $\|e\|$ are comparable in size.
- If $A$ is ill conditioned then $\|r\|$ can be small while $\|e\|$ is large.
- $det(A)$ is not the right way to quantify this connection.
Condition number

**Definition**

Let $\| \cdot \|$ be a matrix norm. Then the condition number of the matrix $A$ induced by the vector norm $\| \cdot \|$ is

$$\text{cond}_{\| \cdot \|}(A) := \|A^{-1}\| \|A\|.$$ 

★ Usually use $\kappa$

$$\kappa(A) = \text{cond}(A) = \text{condition number of } A.$$
Matlab

- `cond(A)` or `cond(A, p)`
- `condest(A)`
- `rcond(A)` = reciprocal condition number
Theorem (Relative Error $\leq \text{cond}(A) \times $ Relative Residual)

Let $Ax = b$ and let $\hat{x}$ be an approximation to the solution $x$. With $r = b - A\hat{x}$

$$\frac{\|x - \hat{x}\|}{\|x\|} \leq \text{cond}(A) \frac{\|r\|}{\|b\|}.$$
Proof.

- $Ae = r$ so $e = A^{-1}r$

\[ \|e\| = \|A^{-1}r\| \leq \|A^{-1}\|\|r\|. \quad (1) \]

- $Ax = b$ so $\|b\| = \|Ax\| \leq \|A\|\|x\|$.

- Dividing the smaller side of (1) by the larger quantity and the larger side of (1) by the smaller gives

\[ \frac{\|e\|}{\|A\|\|x\|} \leq \frac{\|A^{-1}\|\|r\|}{\|b\|}. \]

- Rearrangement proves the theorem.
$\| I \| = 1$ so $\text{cond}(I) = 1$ in any \textit{induced} matrix norm. Similarly for an orthogonal matrix $\text{cond}_2(O) = 1$.

Example: error = $(\text{cond}(A))*\text{residual}$
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Example (The Hilbert Matrix)
The $N \times N$ Hilbert matrix $H_{N \times N}$ is the matrix with entries

$$H_{ij} = \frac{1}{i + j - 1}, \quad 1 \leq i, j \leq n.$$ 

This matrix is extremely ill conditioned even for quite moderate values on $n$. 

$$
\begin{bmatrix}
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\
\frac{1}{2} & 1 & \frac{1}{4} & \frac{1}{5} \\
\frac{1}{3} & \frac{1}{4} & 1 & \frac{1}{6} \\
\frac{1}{4} & \frac{1}{5} & \frac{1}{6} & 1
\end{bmatrix}
$$
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[Basic Result of Backward Error Analysis] The result $\hat{x}$ of solving $Ax = b$ by Gaussian elimination in finite precision arithmetic subject to rounding errors is precisely the same as the exact solution of a perturbed problem

$$(A + E)\hat{x} = b + f$$

(2)

where

$$\frac{\|E\|}{\|A\|}, \frac{\|f\|}{\|b\|} = O(\text{machine precision}).$$
Theorem (Effect of Storage errors in $A$)

Let $A$ be an $N \times N$ matrix. Suppose $Ax = b$ and $(A + E)\hat{x} = b$. Then,

$$\frac{\|x - \hat{x}\|}{\|\hat{x}\|} \leq \text{cond}(A) \frac{\|E\|}{\|A\|}.$$ 

Theorem (Effect of Storage Errors in $b$)

Let $Ax = b$ and $A\hat{x} = b + f$. Then

$$\frac{\|x - \hat{x}\|}{\|x\|} \leq \text{cond}(A) \frac{\|f\|}{\|b\|}.$$
Sketch proof Theorem 145

Proof.
Step 1: By subtraction get an equation for the error driven by the perturbation:

\[ Ax = b \]
\[ (A + E)\hat{x} = b \]

___subtract___

\[ A(x - \hat{x}) = E\hat{x} \]
\[ x - \hat{x} = A^{-1}E\hat{x} \]

Step 2: Take norms

\[ ||x - \hat{x}|| = ||A^{-1}E\hat{x}|| \leq ||A^{-1}|| \cdot ||E|| \cdot ||\hat{x}|| \frac{||A||}{||A||} \]

Step 3: rearrange using def of cond no

\[ \frac{||x - \hat{x}||}{||\hat{x}||} \leq cond(A) \frac{||E||}{||A||}. \]
Sketch proof Theorem 146

Proof.

- Since $A\hat{x} = Ax + f$, $x - \hat{x} = -A^{-1}f$

- $\|x - \hat{x}\| \leq \|A^{-1}\| \|f\| = \text{cond}(A) \frac{\|f\|}{\|A\|}$

- $\|x - \hat{x}\| \leq \text{cond}(A) \frac{\|f\|}{\|b\|} \|x\|$ because $\|b\| \leq \|A\| \|x\|$. 

$\square$
Facts about condition number

▶ When $E$ is due to roundoff errors $\|E\|/\|A\| = O(\text{machine precision})$. \textit{cond}(A) \textbf{tells you how many significant digits are lost (worst case) when solving} $Ax = b$

▶ $\text{cond}(A) \geq 1$ and $\text{cond}(I) = 1$.

▶ Scaling $A$ does not influence $\text{cond}(A)$:

▶ $\text{cond}(A)$ depends on the norm chosen but usually it is of the same order of magnitude for different norms.

▶ If $A$ is symmetric then

$$\text{cond}_2(A) = |\lambda|_{max}/|\lambda|_{min}.$$ 

▶ If $A$ is symmetric, positive definite and $\| \cdot \| = \| \cdot \|_2$, then $\text{cond}(A)$ equals the \textbf{spectral condition number}, $\lambda_{max}/\lambda_{min}$

$$\text{cond}_2(A) = \lambda_{max}/\lambda_{min}.$$ 

▶ $\text{cond}(A) = \text{cond}(A^{-1})$. 
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Lemma (Spectral localization)

For any $N \times N$ matrix $B$ and $\| \cdot \|$ any matrix norm:

$$|\lambda(B)| \leq \|B\|.$$ 

Proof.

For an eigenpair $(\lambda, \phi)$, $B\phi = \lambda\phi$.

Thus $|\lambda|\|\phi\| = \|B\phi\| \leq \|B\|\|\phi\|$.
Theorem 149

**Theorem**

\[ \lim_{n \to \infty} B^n = 0 \text{ if and only if there exists a norm } \| \cdot \| \text{ with } \|B\| < 1. \]

**Proof.**

- We prove \( \|B\| < 1 \Rightarrow \|B^n\| \leq \|B\|^n \to 0 \)
- Start the induction with \( n = 2, \)

\[ \|B^2\| = \|B \cdot B\| \leq \|B\| \cdot \|B\| \leq \|B\|^2. \]

- Assume \( \|B^n\| \leq \|B\|^n, \) then

\[ \|B^{n+1}\| = \|B^n B\| \leq \|B^n\| \|B\| \leq \|B\|^{n+1} \]

- \( \|B^n\| \leq \|B\|^n \to 0 \) as \( n \to \infty. \)