1 Introduction

These theorems are fairly complicated, especially in their proofs, so I will start with an example of the one-dimensional case and then move to higher dimensions. First I want to recall some terminology about functions.

Suppose that $f: \mathbb{R}^n \to \mathbb{R}^m$ and suppose $W \subset \mathbb{R}^m$. We define the “inverse image of $W$ under $f$” as follows:

$$ f^{-1}(W) := \{ x \in \mathbb{R}^n | f(x) = y \text{ for some } y \in W \}. $$

This definition does not require that the function $f$ has an inverse. There might be more than one $x$ for each $y$. As an example, if $n = m = 1$, $f(x) = x^2$ and $W = (0, 1)$, then $f^{-1}(W) = (-1, 1)$. The function $f$ is not 1:1 on $(-1, 1)$ and so doesn’t have an inverse there. However, the function $f|_{(0,1)}$ is 1:1 on $(0,1)$ and does have an inverse, which is $g(y) = \sqrt{y}$ for $0 < y < 1$.

Now recall Theorem 4.1.4 on page 180. One consequence of this theorem is the following:

**Lemma 1** Suppose that $f: \mathbb{R}^n \to \mathbb{R}^m$. Then $f$ is continuous if and only if $f^{-1}(W)$ is an open set for each open subset $W$ of $\mathbb{R}^m$.

As examples, we can consider again $f(x) = x^2$ and notice that $f^{-1}(0,1) = (-1,1)$. On the other hand, suppose that

$$ f(x) = \begin{cases} x & \text{if } x \leq \frac{1}{2} \\ -x & \text{if } x > \frac{1}{2} \end{cases} $$

Let $W = (\frac{1}{4}, \frac{3}{4})$. Then $f^{-1}(W) = (\frac{1}{4}, \frac{1}{2}] \cup (\infty, -\frac{1}{2})$. This is not an open set.

2 Inverse function theorem in one variable

**Theorem 2** Suppose $f: \mathbb{R} \to \mathbb{R}$ and $f \in C^1$. (That is, $f'(x)$ exists for each $x \in \mathbb{R}$ and $f'$ is continuous.) Suppose that $f(a) = b$ and $f'(a) \neq 0$. Then there are open neighborhoods $U$ of $a$ and $W$ of $b$ such that
1. \( f ( U ) = W \) and \( f | U \) is one to one.

2. If \( h \) is the inverse of \( f | U \), then \( h \in C^1 \).

3. For each \( x \in U \), \( h' ( f ( x ) ) = \frac{1}{f' ( x )} \).

**Proof.** We will first prove the theorem under the assumptions that \( a = b = 0 \) and \( f' ( 0 ) = 1 \). In this case, let

\[
G ( x ) = x - f ( x ).
\]

Then \( G ( 0 ) = G' ( 0 ) = 0 \). Since \( f' \) is continuous, \( G \in C^1 \), and so there is an \( \varepsilon > 0 \) such that if \(-2\varepsilon \leq x \leq 2\varepsilon \), then \( |G' ( x )| \leq \frac{1}{2} \). Let

\[
W = ( -\varepsilon , \varepsilon ),
\]

and

\[
U = f^{-1} ( W ) \cap ( -2\varepsilon , 2\varepsilon ) = \{ x \in ( -2\varepsilon , 2\varepsilon ) \mid f ( x ) \in W \}.
\]

Then \( f ( U ) = W \). \( W \) is open, and since the inverse image of continuous function is open, \( U \) is open as well. (\( U \) is the intersection of two open sets.) We must show that \( f | U \) is 1 : 1. This is pretty easy to prove, but we will give a proof which can be generalized to the \( n \)-dimensional case.

For each \( y \in W \), let \( g_y ( x ) = y + G ( x ) \). Then \( g_y \in C^1 \), and \( g'_y ( x ) = G' ( x ) \). Hence, if \( x \in [ -2\varepsilon , 2\varepsilon ] \), then \( |g'_y ( x )| \leq \frac{1}{2} \). This shows that \( g_y \) is a contraction on \( ( -2\varepsilon , 2\varepsilon ) \).

We want to apply the contraction mapping theorem, but to do this we need to show that \( g_y : ( -2\varepsilon , 2\varepsilon ) \to ( -2\varepsilon , 2\varepsilon ) \). If \( x \in ( -2\varepsilon , 2\varepsilon ) \), then

\[
|g_y ( x )| = |y + G ( x )| \leq |y| + |G ( x )|.
\]

Since \( y \in W \), \( |y| < \varepsilon \). Since \( |G' ( c )| \leq \frac{1}{2} \) for \( c \in ( -2\varepsilon , 2\varepsilon ) \), and \( G ( 0 ) = 0 \), the mean value theorem implies that \( |G ( x )| \leq \frac{1}{2} |x| \) for \( x \in ( -2\varepsilon , 2\varepsilon ) \). Hence,

\[
|g_y ( x )| < \varepsilon + \varepsilon = 2\varepsilon.
\]

Therefore \( g_y \) is a contraction and maps \( ( -2\varepsilon , 2\varepsilon ) \) into itself. Hence, there is a unique \( x \in ( -2\varepsilon , 2\varepsilon ) \) such that

\[
y + x - f ( x ) = x,
\]

which implies that \( f ( x ) = y \). On the other hand, if \( f ( x ) = y \), then \( x \) is a fixed point for \( g_y \).

We get such an \( x \) for each \( y \in W \). Define \( h : W \to ( -2\varepsilon , 2\varepsilon ) \) by letting \( h ( y ) \) be the unique fixed point of \( g_y \) in \( ( -2\varepsilon , 2\varepsilon ) \). The uniqueness of \( h ( y ) \) implies that
\[ f|_{(-2\varepsilon, 2\varepsilon)} = 1 : 1, \text{ and since } U \subset (-2\varepsilon, 2\varepsilon), \ f|_U \text{ is } 1 : 1 \text{ and } h = (f|_U)^{-1}. \] We have left to show that \( h \) is differentiable and \( h'(y) = \frac{1}{f'(h(y))} \).

We first show that \( h \) is continuous. Suppose \( y_1 \) and \( y_2 \) are in \( W \), and let \( x_1 = h(y_1), x_2 = h(y_2). \) Since \( x = f(x) + G(x) \), we obtain that

\[
|x_1 - x_2| = |f(x_1) - f(x_2) + G(x_1) - G(x_2)| \leq |f(x_1 - f(x_2))| + \frac{1}{2} |x_1 - x_2|,
\]
or

\[
\frac{1}{2} |x_1 - x_2| \leq |f(x_1) - f(x_2)|.
\]

In other words,

\[
|h(y_1) - h(y_2)| \leq 2 |y_1 - y_2|,
\]
showing that \( h \) is continuous.

Next note that on \((-2\varepsilon, 2\varepsilon)\), \( f' = 1 - G' \), and since \( |G'(x)| \leq \frac{1}{2} \), \( f'(x) \geq \frac{1}{2} \). In particular, \( f'(h(y)) \neq 0 \). To show that \( h \) is differentiable we use the definition of derivative, and compute

\[
\lim_{y \to y_0} \frac{h(y) - h(y_0) - \frac{1}{f'(h(y_0))} (y - y_0)}{|y - y_0|}
\]
for some \( y_0 \in (-\varepsilon, \varepsilon) \). If \( x \) and \( x_0 \) are in \((-2\varepsilon, 2\varepsilon)\), and \( h(x) = y, h(x_0) = y_0 \), then

\[
\frac{|h(y) - h(y_0) - \frac{1}{f'(h(y_0))} (y - y_0)|}{|y - y_0|} = \frac{|x - x_0| - \frac{1}{f'(x_0)} (f(x) - f(x_0))}{|f(x) - f(x_0)|}.
\]

Applying the mean value theorem to \( f(x) - f(x_0) \) in both the numerator and denominator,

\[
\frac{|h(y) - h(y_0) - \frac{1}{f'(h(y_0))} (y - y_0)|}{|y - y_0|} = \frac{|x - x_0| - f'(c)(x - x_0)}{|f'(c)(x - x_0)|} \cdot \frac{|f'(c)|}{|f'(x_0)|} \left(1 - \frac{f'(c)}{f'(x_0)}\right),
\]
where \( c \) is between \( x \) and \( x_0 \). From (1) the first term is bounded by 2. Since \( h \) is continuous, \( \lim_{y \to y_0} (x - x_0) = \lim_{y \to y_0} (h(y) - h(y_0)) = 0. \) Since \( f' \) is continuous and \( c \) is between \( x \) and \( x_0 \), the second term tends to zero as \( x \to x_0 \), proving that \( h'(y_0) \) exists and that \( h'(y_0) = \frac{1}{f'(h(y_0))} \). This proves Theorem 2 in the case where \( f(0) = 0 \) and \( f'(0) = 1 \). \( \blacksquare \)

There are three steps to complete the proof of the theorem.
1. We remove the requirement that \( f'(0) = 1 \), and only require that \( f'(0) \neq 0 \). If \( f'(0) = \alpha \), let \( g(x) = f(x/\alpha) \). Then \( g'(0) = 1 \). Prove the theorem for \( g \) and it easily gives a comparable theorem for \( f \).

2. Suppose this is done, and we only require that \( f'(0) \neq 0 \). We now remove the requirement that \( f(0) = 0 \). Introduce a new \( g : g(x) = f(x) - f(0) \). Then \( g(0) = 0 \), and the theorem for \( g \) implies the theorem for \( g \).

3. To remove the requirement that \( a = b = 0 \), let \( z = x - a \) and \( h(z) = f(z + a) - b \). then \( h(0) = 0 \) and we are back to a previous case.

3  Inverse function theorem for \( f : \mathbb{R}^n \to \mathbb{R}^n \).

See that auxiliary notes Notes 6a, which follow the exercises below, where a side by side comparison is given of the one dimensional and \( n \)-dimensional cases. This is restricted to the case of \( f(0) = 0 \), \( Df(0) = I \) (the identity transformation). Steps similar to 1,2,3 above give the general case, in which it is assumed that \( Df(x_0) \) is a 1 : 1 map; that is \( Df(x_0)(u) = 0 \) if and only if \( u = 0 \). This is equivalent saying that the standard matrix of \( Df(x_0) \) (which is \( n \times n \)) is non-singular. We then get the theorem in the text (pg. 393), which I will not copy here.

4  Implicit function theorem.

The following is another way of stating the inverse function theorem:

**Theorem 3** Suppose that \( f : \mathbb{R}^n \to \mathbb{R}^n \) satisfies the conditions of the inverse function theorem (as given in notes 6a). Suppose that \( f(x_0) = y_0 \). Then in a neighborhood \( U \times W \) of the point \( (x_0, y_0) \) of \( \mathbb{R}^n \times \mathbb{R}^n \), the equations

\[
 f_i(x_1, \ldots, x_n) = y_i, \quad \text{for } i = 1, \ldots, n \tag{2}
\]

can be solved for \( (x_1, \ldots, x_n) \) in terms of \( (y_1, \ldots, y_n) \).

Note that (2) is a system of \( n \) equations for the \( n \) unknowns \( x_1, \ldots, x_n \). For any \( y \in W \), the unique solution \( x \) of these equations in \( U \) is \( x = f^{-1}(y) \), where \( f^{-1} \) is the function \( h \) found in the inverse function theorem.

The one dimensional version of this is particularly simple: The implicit function theorem is a generalization of this.
Theorem 4 Suppose that $F : R^n \times R^m \rightarrow R^m$, and $F \in C^1$. Suppose the $F(x_0, y_0) = 0 \in R^m$, for some $(x_0, y_0) \in R^n \times R^m$. Let

$$\Delta = \det \begin{pmatrix} \frac{\partial F_1}{\partial y_1}(x_0, y_0) & \frac{\partial F_1}{\partial y_2}(x_0, y_0) & \cdots & \frac{\partial F_1}{\partial y_m}(x_0, y_0) \\ \frac{\partial F_2}{\partial y_1}(x_0, y_0) & \frac{\partial F_2}{\partial y_2}(x_0, y_0) & \cdots & \frac{\partial F_2}{\partial y_m}(x_0, y_0) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial y_1}(x_0, y_0) & \frac{\partial F_m}{\partial y_2}(x_0, y_0) & \cdots & \frac{\partial F_m}{\partial y_m}(x_0, y_0) \end{pmatrix}.$$  

Assume that $\Delta \neq 0$. Then there are open neighborhoods $U$ of $x_0$ in $R^n$ and $V$ of $y_0$ in $R^m$, and a unique function $f : U \rightarrow V$ such that $F(x, f(x)) = 0$ for all $x \in U$. Further, $f$ is differentiable at each $x \in U$.

We will only prove the theorem in the case $n = 3, m = 2$, in which case we can write out everything explicitly. Before giving the proof, we give examples in this case. The easiest examples are linear, such as

$$\begin{align*}
x + y + z &= u + v \\
x - y - z &= u - v
\end{align*} \tag{3}$$

In this case,

$$F(x, y, z, u, v) = \begin{pmatrix} x + y + z - u - v \\ x - y - z - u + v \end{pmatrix},$$

and (3) is equivalent to $F(x, y, z, u, v) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. We see that

$$\Delta = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix},$$

so $\det \delta = -2$. We can solve for $u$ and $v$ in terms of $x, y, z$ by adding and subtracting the equations, which give

$$\begin{align*}
u &= x \\
v &= \frac{1}{2} (y + z).
\end{align*}$$

We can set $U = R^3, V = R^2$, since this solution is valid for all $(x, y, z)$.

Now change the equations a bit:

$$\begin{align*}
x + y + z &= u^2 + v \\
x - y - z &= u - v \tag{4}
\end{align*}$$
Adding the equation gives
\[ u^2 + u = 2x \]
We can solve this for \( u \) in terms of \( x \) by using the quadratic formula:
\[ u^2 + u - 2x = 0 \]
\[ u = \frac{-1 \pm \sqrt{1 + 8x}}{2}. \]
Then we can solve for \( v \):
\[ v = \frac{-1 \pm \sqrt{1 + 8x}}{2} - (x - y - z). \] (5)

We see that the solution is not unique. This is where neighborhoods come in. Looking at the statement of the theorem, we see that we need to pick a point \((x_0, y_0)\) which is a solution of the set of equations. First, though, let’s write down the function \( F \):
\[ F(x, y, z, u, v) = \begin{pmatrix} x + y + z - u^2 - v \\ x - y - z - u + v \end{pmatrix}. \]
Next we need to pick a specific point in \( R^3 \times R^2 \) (the point called \((x_0, y_0)\) in the theorem). For instance, we can chose the point \((0, 0, 0, 0) \in R^3 \times R^2\). We see that \( F(0, 0, 0, 0, 0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \). Now we find \( \Delta \):
\[ \Delta = \det \begin{pmatrix} -2u & -1 \\ -1 & 1 \end{pmatrix} \big|_{(0,0,0,0,0)} = -1. \]
This is nonzero, so the theorem applies. We can solve for \((u, v)\) in terms of \((x, y, z)\), and indeed, we did so. But to get a unique solution we need to choose neighborhoods. The point \( x_0 \) is now \((0, 0, 0)\), and the point \( y_0 \) is \((0, 0)\). The solution \( y = f(x) \) is now written \((u, v) = f(x, y, z)\), and we must have \((0, 0) = f(0, 0, 0)\). Notice in (5) and the equation just above, that if we choose the minus sign, we get \( f(0, 0, 0) = (-1, -1) \). Therefore we must choose the plus sign. We get
\[ f(x, y, z) = \begin{pmatrix} \frac{-1 + \sqrt{1 + 8x}}{2} \\ \frac{-1 + \sqrt{1 + 8x}}{2} - (x - y - z) \end{pmatrix}. \]
This doesn’t make sense of \( x < -\frac{1}{8} \). And the solution to (4) is not unique if we allow \( U \times W \) to include the point \((-1, -1, 0, 0, 0)\). For neighborhoods we could choose
\[ U = \left\{ (x, y, z) \mid -\frac{1}{8} < x \right\}, \quad W = \left\{ (u, v) \mid u > -\frac{1}{2} \right\} \]
If we take the alternate sign in the solution, the minus sign, then we obtain $u < -\frac{1}{2}$. So in $U$ there is a unique solution for $(u, v)$ which lies in $W$.

5 Homework due February 16

1. Suppose $f : \mathbb{R}^2 \to \mathbb{R}^1$. Suppose that $f \in C^1$. Can $f|_U$ have an inverse for some open set $U$? Can $f|_U$ have a continuous inverse? Can $f|_U$ have a differentiable inverse?

2. pg. 396, # 1.

3. pg. 396, # 2

4. pg. 400, # 1

5. pg. 401, # 5.