1 Pullback on $R^2$

We now turn to a more subtle proof of Green’s theorem, which generalizes more easily to larger $k$ and $n$. The basic idea is to somehow use the proof for the unit square to cover a general 2-cell $\phi$, by what is called a “pullback”.

(This is still following the book by Pugh, except for notation in some places.)

Recall that we let $S_{k,2}$ denote the set of $k$-cells on $R^2$. Only $k = 0, 1, 2$ make sense on $R^2$. Let $Y$ denote a functional mapping $S_{k,2}$ into $R$, for some $k$. $Y$ could be a $k$-form, but it doesn’t have to be. For example, if $k = 1$, then $Y(\phi)$ could be the length of $\phi$.

**Definition 1** Let $T : R^2 \rightarrow R^2$ be a smooth transformation. Then the “pullback” $T^*$ defined by $T$ is a mapping from a functional $Y$ on $S_{k,2}$ to another functional $T^*Y$ on $S_{k,2}$ defined by

$$T^*Y(\phi) = Y(T \circ \phi)$$

for any $k$-cell $\phi$ on $R^2$. Note that this makes sense for any $k = 0, 1, 2$, because $\phi : I^k \rightarrow R^2$ and $T : R^2 \rightarrow R^2$.

In our applications to Green’s and Stokes’ theorems, $Y$ will be a differential form. The same $T$ defines a pullback operation on 0-forms, 1-forms and 2-forms on $R^2$.

**Example 2** Let $\omega = F : R^2 \rightarrow R^2$, a 0-form. Let $\phi = (x_0, y_0)$, a 0-cell. Then $T^*\omega(\phi) = F(T(x_0, y_0))$. In other words, $T^*\omega = F \circ T$.

**Example 3** Let $\omega = dx$, a basic 1-form on $R^2$. We have seen that if $\phi = (x(t), y(t))$ is any 1-cell, then

$$\omega(\phi) = x(1) - x(0),$$

the net change in $x$ on the curve $\phi$.

Let $T(x, y) = \left(\frac{1}{2}x \cos \pi y, \frac{1}{2}x \sin \pi y \right)$. Then

$$(T \circ \phi)(t) = \left(\frac{1}{2}x(t) \cos \pi y(t), \frac{1}{2}x(t) \sin y(t) \right),$$

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and
\[
\omega(T \circ \phi) = \frac{1}{2}x(1) \cos \pi y(1) - \frac{1}{2}x(0) \cos \pi y(0).
\]

For instance, suppose that \(\phi(t) = (2t, 4t)\) for \(0 \leq t \leq 1\). Then \((T \circ \phi)(t) = (t \cos 4\pi t, t \sin 4\pi t)\). The image of \(T \circ \phi\) is quite different from the image of \(\phi\), as shown below, but \(T^* \omega(\phi) = \omega(T \circ \phi)\) is still the change in \(x\) along the transformed curve, which in this case, and usually, is different from the change in \(x\) along \(\phi\).

**Example 4** Let \(\omega = dx \wedge dy\), a 2-form. We have seen that if \(\phi\) is a \(1:1\) 2-cell, then \(\omega(\phi)\) is \(\pm\) the area of \(\phi(I^2)\). Then, \(T^* \omega(\phi) = \omega(T \circ \phi)\) is \(\pm\) the area of \(T(\phi(I^2))\) if \(T \circ \phi\) is \(1:1\) on \(I^2\). For instance, if \(\phi(x, y) = (2x, 3y)\), and \(T(x, y) = (x + y, y^3)\), then both \(\phi\) and \(T\) are \(1:1\), and so \(T \circ \phi\) is \(1:1\). We see that

\[
T \circ \phi(x, y) = (2x + 3y, 27y^3).
\]

The image \(\phi(I^2)\) is a rectangle with area 6. To find the image of \(T \circ \phi\), find its boundary by looking at the images of the boundary of \(I^2\). We see that the boundary of \(T \circ \phi(I^2)\) consists of the four curves \(v = u^3\), \(v = (u - 2)^3\), \(v = 0\), \(v = 27\). We then have that

\[
T^* \omega(\phi) = \omega(T \circ \phi) = \int_{T \circ \phi} dx \wedge dy
= \int_0^1 \int_0^1 J(T \circ \phi) \, dxdy = \int_0^1 \int_0^1 \det \left( \begin{array}{cc} 2 & 3 \\ 0 & 81y^2 \end{array} \right) \, dxdy
= \int_0^1 162y^2 \, dy = 54.
\]

Observe that if \(\phi\) is a \(k\)-cell, then \(T \circ \phi\) is also a \(k\)-cell. The key lemma in the proof of Green’s theorem is the following:

**Lemma 5** The pullback operation has the following properties:

1. The transformation \(T \rightarrow T^*\) is linear, and \((S \circ T)^* = T^* \circ S^*\)
2. The pullback of a $k$-form is a $k$-form. In particular, for the 2-form $dx \wedge dy$ we have
\[ T^* (dx \wedge dy) = dT_1 \wedge dT_2, \] (2)
where $T_1$ and $T_2$ are the components of $T$. (Note that $T_1$ and $T_2$ are 0-forms, and so $dT_1$ and $dT_2$ are 1-forms. Hence $dT_1 \wedge dT_2$ is a 2-form)

3. $T^* (\alpha \wedge \beta) = T^* (\alpha) \wedge T^* (\beta)$. Here $\alpha$ could be a $k$-form and $\beta$ a $j$-form with $i + j \leq 2$.

4. If $\omega$ is a 0-form or a 1-form, then $d(T^* \omega) = T^* (d\omega)$. In other words, $T^*$ and $d$ commute: $T^* d = d T^*$.

We will defer the proof of this lemma until the next set of notes. We skip right to the proof of Green’s theorem.


Proof. In this proof, $T = \phi$, Observe that everything above, including the Lemma, hold for any pair $(T, \phi)$ of smooth functions $T : U \to \mathbb{R}^2$ and $\phi : I^2 \to \mathbb{R}^2$ such that $\phi(I^2) \to U$. This means that $T \circ \phi : I^2 \to \mathbb{R}^2$ is a $k$-cell, and all of the results above hold. (Those results which involve $T_1$ and $T_2$ require that both are defined on the image of $\phi$.)

To prove Green’s theorem, we let $\xi$ be any 2-cell, and $i : I^2 \to \mathbb{R}^2$ be the identity map $i(x, y) = (x, y)$ for $(x, y) \in I^2$, and then in the formulas and results above we let $T = \xi$ and $\phi = i$. Note then that $T \circ \phi = \xi \circ i$ is defined, for any 2-cell $\xi$. Let $\omega$ be any 1-form on $\mathbb{R}^2$. We then have the following string of equalities:
\[ \int_{\xi} d\omega = \int_{\xi \circ i} d\omega \] (3)
(because $\xi \circ i = \xi$)
\[ = \int_{i} \xi^* d\omega \] (4)
(from the definition of $\xi^* d\omega$)
\[ = \int_{i} d (\xi^* \omega) \] (5)
(from (4) in the Lemma)

\[ = \int_{\partial i} \xi^* \omega \]  \hspace{1cm} (6)

from Green’s theorem for the unit square

\[ = \int_{\xi \circ \partial i} \omega \]  \hspace{1cm} (7)

from the definition of $\xi^* \omega$ , where here $\xi \circ \partial i$ is the chain obtained by composing $\xi$ with the second term of each element $\pm 1, (\partial i)_j$ of the chain $\partial i = \int_{\partial \xi} \omega$ from the definition of boundary the boundary of $\xi$, since, in fact, each term $(\partial i)_j$ is the identity on some side of $I^2$. To be more precise about this last step, note for example that $i (1, t) = (1, t)$ , and therefore $\xi (1, t) = \xi (i(1, t)) = (\xi \circ i)(1, t)$.

This prove’s Greens’ theorem for any 2-cell $\xi$. ■

3  Homework due April 13

1. Find a 1-form $\omega$ such that if $\phi$ is a $1:1$ 2-cell, then $\int_{\partial \phi} \omega$ is plus or minus the area of the image of $\phi$. Use this 1-form to evaluate the area inside the ellipse $ax^2 + by^2 = 1$, where $a > 0, b > 0$.

2. (20 pts.) Let $\xi (x, y) = (x + 2y, 3x - y)$. Let $\omega = ydx$ on $R^2$. The final proof of Green’s theorem above involves six equations, (3)-(7). For this particular pair of a 2-cell and a 1-form, work out each of the seven integrals in the six steps of the proof from the definitions of these quantities. In each case, your answer should be a number – the same number! Hint: Several steps are very easy. One that requires thought involves figuring out $\xi^* \omega$. (You have to find that before you can find $d(\xi^* \omega)$.) This is a 1-form $pdx + qdy$. You have to find $p$ and $q$. Another step that involves some thinking is to find $\xi \circ \partial i$. Do this one element at a time. It is defined just after (7).