1 Which sets have volume 0?

The theorem at the end of the last section makes this an important question. (Measure theory would supersede it, however.)

**Theorem 1** A set $A \subset \mathbb{R}^n$ has volume zero if and only if for every $\varepsilon > 0$ there is a finite covering of $A$ by rectangles with total volume less than $\varepsilon$.

This is worked example 8.1 on page 487, so I will omit the proof in these notes. The “only if” part was a lemma in notes #9.

(Note the difference between this and the definition of a set of measure 0. In the latter case, you can allow an infinite covering, so long as it is countable.)

We can now list some sets with zero volume, and ask some interesting questions about volume zero and measure zero.

1. A finite set of points. This should be obvious; if there are $m$ points, and $\varepsilon > 0$, find a box of volume less than $\frac{\varepsilon}{m}$ containing each point.

2. The image of a smooth curve $\gamma : [a, b] \to \mathbb{R}^n$. (This is implied by a homework problem, which is more difficult than most.)

3. What about the image of a continuous curve $\gamma : [a, b] \to \mathbb{R}^n$?

4. What about the image of $f(x) = (x, \sin \frac{1}{x})$ over $(0, 1)$?

5. (see page 454, # 5 and 6): Must the boundary of a set have measure zero? Must the boundary of a set of measure zero have measure zero? Suppose that $A$ has no volume. Can its boundary $\partial A$ have volume?
2 Properties of the integral

These are listed in Theorem 8.4.1. They all follow from a similar property for sums. The one needing discussion is:

**Theorem 2** Mean Value Theorem for Integrals: Suppose that $f: A \to \mathbb{R}$, with $A \subset \mathbb{R}^n$ bounded, $f$ is continuous, and the boundary of $A$ has volume zero. Suppose that $A$ is compact and connected. Then $\int_A f$ exists, and there is an $x_0 \in A$ such that

$$\int_A f = f(x_0) V(A).$$

Proof: The last theorem of the previous notes shows that $\int_A f$ exists.

To get the mean value theorem, let $m = \inf \{ f(x) \mid x \in A \}$ and $M = \sup \{ f(x) \mid x \in A \}$. Then by the definitions of upper and lower sums, we see that $m V(A) < \int_A f < M V(A)$, or $m < \frac{\int_A f}{V(A)} < M$. Because $A$ is compact and $f$ is continuous, $m$ and $M$ are values taken on by $f$ in $A$. Hence by the intermediate value theorem for a continuous function on a connected set, there is an $x_0 \in A$ such that $f(x_0) = \frac{\int_A f}{V(A)}$, which implies the theorem.

3 Improper integrals

We have only defined $\int_A f$ in the case where $A$ is bounded and $f$ is bounded. If either of these conditions is not satisfied, then $\int_A f$ has not been defined. We first deal with the case where $A$ is unbounded. We also assume that $f$ is nonnegative.

**Definition 3** Suppose that $A \subset \mathbb{R}^n$ and $f: A \to \mathbb{R}$ and $f$ is bounded. Suppose also that $f(x) \geq 0$ for all $x \in A$. Suppose that for every $n$, $B_n = [-n, n] \times [-n, n] \times \cdots \times [-n, n]$. (This will be called the “$n$-cube”.) Suppose that for each $n$, $\int_{A \cap B_n} f$ exists. Suppose finally that

$$\lim_{n \to \infty} \int_{A \cap B_n} f$$

exists. Then we define $\int_A f$ to be this limit.

Next we consider the case where $f$ may be unbounded. We still assume that $f \geq 0$. In this case, we let

$$f_M(x) = \begin{cases} f(x) & \text{if } x \in A \text{ and } f(x) \leq M \\ 0 & \text{if } f(x) > M. \end{cases}$$
Definition 4 If \( f_M \) is integrable over \( A \) for each \( M \), and \( \lim_{M \to \infty} \int_A f_M \) exists, then we define \( \int_A f \) to be this limit.

When we say “integrable over \( A \)” in this definition, we mean according to the previous definition, which allows \( A \) to be unbounded.

Finally, we remove the requirement that \( f \geq 0 \). Let

\[
f^+ (x) = \begin{cases} 
  f (x) & \text{if } x \in A \text{ and } f (x) \geq 0 \\
  0 & \text{if } x \in A \text{ and } f (x) < 0
\end{cases}
\]

and

\[
f^- (x) = \begin{cases} 
  -f (x) & \text{if } x \in A \text{ and } f (x) \leq 0 \\
  0 & \text{if } x \in A \text{ and } f (x) > 0
\end{cases}
\]

Then for any \( x \in A \), \( f (x) = f^+ (x) - f^- (x) \). We define \( \int_A f \) to be \( \int_A f^+ - \int_A f^- \), assuming both of these exist according to the definitions above.

3.1 The last definition is important!

This definition leads to an interesting conclusion. Let \( A = (-\infty, \infty) \subset R \), and on \( A \), let \( f (x) = x \). Does \( \int_A f \) exist? You might think so, because when \( n = 1 \), \( B_n = [-n, n] \), and

\[ \int_{-n}^{n} f \big|_{-n,n} = 0 \]

for each \( n \). But

\[ \int_A f^+ = \lim_{n \to \infty} \int_{-n}^{n} f^+ = \int_{0}^{n} f^+ = \frac{1}{2} n^2, \]

and so \( \int_A f^+ \) does not exist. Therefore, \( \int_A f \) does not exist.

3.2 An example in \( R^2 \).

As far as I have seen, the text gives no examples, at least in this chapter, of improper integrals which exist when \( n > 1 \). Here is an example for \( n = 2 \).

Let \( A = \{(x, y) \mid x^2 + y^2 \leq 1\} \). For \( (x, y) \in A \), let

\[ f (x, y) = \begin{cases} 
  \frac{1}{(x^2+y^2)^{1/2}} & \text{if } (x, y) \neq (0,0) \\
  0 & \text{if } (x, y) = (0,0)
\end{cases}. \]
To show that this integral exists I will jump ahead in the book and use the theory of “change of variables” in sections 9.3 and 9.4, which shows that this integral exists if and only if
\[ \int_{A'} g \]
exists, where \( A' \) is the rectangle in \( \mathbb{R}^2 \) given by
\[ A' = \{(r, \theta) \mid 0 \leq r \leq 1, \ 0 \leq \theta \leq 2\pi \} , \]
and in \( A' \),
\[ g(r, \theta) = \begin{cases} \frac{1}{\sqrt{r}} & \text{if } r \neq 0 \\ 0 & \text{if } r = 0. \end{cases} \]
(Recall from calc III that you do an integral in polar coordinates by setting \( \iint f(x, y) \, dx \, dy = \int \int f(r \cos \theta, r \sin \theta) r \, dr \, d\theta \). Then \( \iint_A \frac{1}{\sqrt{x+y^2}} \, dx \, dy = \int \int_{A'} \frac{1}{\sqrt{r^2}} \, r \, dr \, d\theta = \int \int_{A'} \frac{1}{\sqrt{r}} \, dr \, d\theta \).
This is all justified in Chapter 9.)

Note that \( g \) does not depend on \( \theta \), and that \( \int_{A'} g \) is an improper integral. You evaluate this integral in the usual way:
\[ g_M (r, \theta) = \begin{cases} \frac{1}{\sqrt{r}} & \text{if } r \geq \frac{1}{M^2} \\ 0 & \text{if } r < \frac{1}{M^2} \end{cases} \]
In chapter 9 the usual theory of iterated integral is developed, as a practical way of evaluating integrals in \( \mathbb{R}^n \). In this case it turns out that
\[ \int_{A'} g_M = \int_0^{2\pi} \left( \int_0^{1/M^2} \frac{1}{\sqrt{r}} \, dr \right) \, d\theta = \pi \left( 4 - 4 \sqrt{\frac{1}{M^2}} \right) . \]
Then we see that \( \lim_{M \to \infty} \int_{A'} g_M = 4\pi \), which is therefore the value of \( \int_{A'} g \) and \( \int_A f \).

This example is interesting, because if \( A_1 \subset \mathbb{R}^2 \) is the \( x \)-axis, then \( f|_{A_1} (x, y) = \frac{1}{|x|^{3/2}} \). But \( \int_{-1}^{1} \frac{1}{|x|^{3/2}} := \lim_{\varepsilon \to 0^+} \left( \int_{-\varepsilon}^{\varepsilon} \frac{1}{|x|^{3/2}} + \int_{-\varepsilon}^{\varepsilon} \frac{1}{|x|^{3/2}} \right) \), which does not exist. So it’s surprising that \( f \) is integrable. In some sense the reason is that the area of a small square, say of side \( \delta \) is \( \delta^2 \), which is (a lot) less than the length of its sides if \( \delta \) is very small. So in a partition, the areas of the squares will be less than the lengths of the partitioning line segments on the \( x \) axis.

This is similar to the example of a “building which can be filled on the inside with paint but is too big to paint on the outside.” I’ll discuss this in class!
3.3 Examples of improper integrals.

There are several examples on page 465. These are all in $R^1$. Here are two of them.

1. $\int_0^1 \log x \, dx$. Use integration by parts:

$$\int_0^1 \log x \, dx = x \log x |_0^1 - \int_0^1 \frac{1}{x} \, dx = \varepsilon \log \varepsilon - (1 - \varepsilon).$$

We use l’Hôpital’s rule:

$$\lim_{\varepsilon \to 0^+} \varepsilon \log \varepsilon = \lim_{\varepsilon \to 0} \frac{\log \varepsilon}{1/\varepsilon} = \lim_{\varepsilon \to 0} \frac{1/\varepsilon}{-1/\varepsilon^2} = \lim_{\varepsilon \to 0} -\varepsilon = 0.$$

So the final answer is $-1$.

2. $\int_2^\infty \frac{1}{\log x} \, dx$:

$$\int_2^M \frac{1}{\log x} \, dx > \int_2^M \frac{1}{x} \, dx$$

which diverges.

3.4 Other types of convergence.

Since improper integrals involve limits, if the limit exists then the integral is called “convergent”.

Definition 5 An integral $\int_A f$ is “absolutely convergent” if $\int_A |f|$ exists.

Definition 6 An integral $\int_A f$ is called “conditionally convergent” if it is convergent but not absolutely convergent.

Example 7 Let $f(x) = \frac{\sin x}{x}$ for $1 \leq x < \infty$. The text shows that $\int_1^\infty f$ is conditionally convergent.

4 Homework, due March 16.

1. Page 454, # 1. A problem further down is related to this.

2. Page 454, # 2.

3. page 457, # 5
4. page 490. #3.

5. pg. 492, #22. (The "Gamma function" is important.)

6. pg. 494, #38. Explain why the function in this problem is integrable, while the characteristic function of the rationals in [0, 1] is not integrable. Hint: A theorem we have studied is relevant.

7. Prove that if \( f : [0,1] \to \mathbb{R} \) is continuous, then the graph of \( f \) is a set of zero volume in \( \mathbb{R}^2 \). (Recall that the graph of \( f \) is \( \{(x,y) \mid 0 \leq x \leq 1 \text{ and } y = f(x)\} \). Hint: I found it helpful to think of this graphically.

8. Prove that if \( \gamma = (\gamma_1, \gamma_2) \) is a smooth curve in \( \mathbb{R}^2 \) (so that \( \gamma : [a,b] \to \mathbb{R}^2 \)), then \( V(\gamma([a,b])) = 0 \). Hint: Boundedness of \( \gamma \) is important. And it would be enough just to have \( \gamma_1 \) continuous. And having \( D\gamma \neq 0 \) is not important. And the solution to #7 is a helpful guide.