Honors Calculus Quiz 2 Solutions 9/5/3

Question 1

Given that \( \lim_{t \to a} f(t) = \lim_{t \to a} g(t) = 25 \), compute the following limits:

- \( \lim_{t \to a} \frac{f^2(t)}{\sqrt{16g(t) + 9f(t)}} \)

Since the denominator is well-behaved in the limit, the limit laws show that we can compute this limit by straightforward substitution:

\[
\lim_{t \to a} \frac{f^2(t)}{\sqrt{16g(t) + 9f(t)}} = \lim_{t \to 0} \frac{25^2}{\sqrt{16(25) + 9(25)}} = 25.
\]

- \( \lim_{t \to a} \frac{\sqrt{f(t)} - \sqrt{g(t)}}{g^2(t) - f^2(t)} \)

Here the numerator and denominator both vanish in the limit, so we first have to do some algebra, before taking the limit:

We factorize the denominator, twice, until we can cancel out the problem terms:

\[
\lim_{t \to 0} \frac{\sqrt{f(t)} - \sqrt{g(t)}}{g^2(t) - f^2(t)} = \lim_{t \to 0} \frac{\sqrt{f(t)} - \sqrt{g(t)}}{(g(t) - f(t))(g(t) + f(t))}
\]

\[
= \lim_{t \to 0} \frac{\sqrt{f(t)} - \sqrt{g(t)}}{(\sqrt{g(t)} - \sqrt{f(t)})(\sqrt{g(t)} + \sqrt{f(t)})(g(t) + f(t))}
\]

\[
= - \lim_{t \to 0} \frac{1}{(\sqrt{g(t)} + \sqrt{f(t)})(g(t) + f(t))}
\]

\[
= - \frac{1}{(5 + 5)(25 + 25)} = - \frac{1}{500}.
\]
Question 2

Find each limit, or if a limit does not exist, say why not.

• \( \lim_{t \to 1} \frac{t^3 - 1}{t^2 - 1} \).

\[
\lim_{t \to 1} \frac{t^3 - 1}{t^2 - 1} = \lim_{t \to 1} \frac{(t - 1)(t^2 + t + 1)}{(t - 1)(t + 1)} = \lim_{t \to 1} \frac{t^2 + t + 1}{t + 1} = \frac{3}{2}.
\]

Note that we can also relate this limit to the derivative of \( t^3 \) at \( t = 1 \), as follows:

\[
\lim_{t \to 1} \frac{t^3 - 1}{(t^2 - 1)} = \lim_{t \to 1} \frac{(t^3 - 1)}{(t + 1)(t - 1)} = \left( \lim_{t \to 1} \frac{1}{t + 1} \right) \left( \lim_{t \to 1} \frac{t^3 - 1}{t - 1} \right) = \frac{1}{2} \left( \lim_{t \to 1} \frac{t^3 - 1}{t - 1} \right).
\]

But the quantity \( \lim_{t \to 1} \frac{t^3 - 1}{(t^2 - 1)} \) is just the derivative of the function \( t^3 \) evaluated at \( t = 1 \), so as shown in class is \( [3t^2]_{t=1} = 3 \).

So the required limit is \( \frac{1}{2} \left[ \frac{d}{dt}(t^3) \right]_{t=1} = \frac{3}{2} \).

• \( \lim_{t \to 2} \frac{1}{(t - 2)} \left( \frac{1}{t^2} - \frac{1}{4} \right) \).

\[
\lim_{t \to 2} \frac{1}{(t - 2)} \left( \frac{1}{t^2} - \frac{1}{4} \right) = \lim_{t \to 2} \frac{1}{(t - 2)} \left( \frac{4 - t^2}{4t^2} \right) = \lim_{t \to 2} \frac{1}{(t - 2)} \left( \frac{(2 - t)(2 + t)}{4t^2} \right) = \lim_{t \to 2} \left( \frac{2 + t}{4t^2} \right) = -\frac{2 + 2}{4(2^2)} = -\frac{1}{4}.
\]

Note that this quantity is the definition of the derivative of \( t^{-2} \) evaluated at \( t = 2 \).

So once we have proved that the power rule formula \( \frac{d}{dt}t^n = nt^{n-1} \) is true for all real constants \( n \), when \( t > 0 \), we may write the answer as:

\[
\lim_{t \to 2} \frac{1}{(t - 2)} \left( \frac{1}{t^2} - \frac{1}{4} \right) = \left[ \frac{d}{dt}(t^{-2}) \right]_{t=2} = [-2t^{-3}]_{t=2} = -2(2^{-3}) = -2^{-2} = -\frac{1}{4}.
\]
\[ \lim_{t \to 25} \frac{5 - \sqrt{t}}{t^2 - 625} \]
\[ = \lim_{t \to 25} \frac{(5 - \sqrt{t})}{(t - 25)(t + 25)} \]
\[ = \lim_{t \to 25} \frac{(5 - \sqrt{t})}{(\sqrt{t} - 5)(\sqrt{t} + 5)(t + 25)} \]
\[ = -\lim_{t \to 25} \frac{1}{(\sqrt{t} + 5)(t + 25)} = -\frac{1}{10(50)} = -\frac{1}{500}. \]

Note that this part is a special case of the second part of question one: take \( a = 25, f(t) = 25 \) and \( g(t) = t \), so it is not surprising that the answer is the same.

Alternatively we can again relate this problem to a suitable derivative, in this case the derivative of the function \(-t^{\frac{1}{2}}\), evaluated at \( t = 25 \), as follows:

\[ \lim_{t \to 25} \frac{(5 - \sqrt{t})}{t^2 - 625} = \lim_{t \to 25} \frac{(5 - \sqrt{t})}{(t + 25)(t - 25)} \]
\[ = \left( \lim_{t \to 25} \frac{1}{t + 25} \right) \left( \lim_{t \to 25} \frac{(5 - \sqrt{t})}{(t - 25)} \right) \]
\[ = \frac{1}{50} \left( \lim_{t \to 25} \frac{((-t^{\frac{1}{2}}) - (-5))}{(t - 25)} \right) \]
\[ = \frac{1}{50} \left( \frac{d}{dt} \frac{d}{dt} (-t^{\frac{1}{2}}) \right)_{t=25} = \frac{1}{50} [\frac{-1}{2} t^{-\frac{3}{2}}]_{t=25} \]
\[ = -\frac{1}{100} (25^{-\frac{3}{2}}) = -\frac{1}{100} \left( \frac{1}{5} \right) = -\frac{1}{500}. \]
Question 3

A particle moves along the $x$-axis, such that its position $x$ in meters from the origin at time $t$ seconds is given by the formula (valid for $t > -1$):

$$x = \frac{1}{t+1} - t.$$ 

- Compute the average velocity of the particle taken over the time interval from 0 to $t$ (where $t \neq 0$ and $t > -1$).

The average velocity $V_{a\to b}$, taken from $t = a$ to $t = b$, with $a \neq b$ is:

$$V_{a\to b} = \frac{\Delta x}{\Delta t} = \frac{x(b) - x(a)}{b - a}.$$ 

Here we want $a = 0$ and $b = t$. Note that $x(0) = 1$. So the average velocity $V(t) = V_{0\to t}$, taken from time 0 to $t$ (where $0 \neq t > -1$) is:

$$V(t) = \frac{x(t) - x(0)}{t - 0} = \frac{\frac{1}{t+1} - 1}{t} = \frac{1 - (t + 1)^2}{t(t + 1)} = \frac{(1 - (t + 1))(1 + (t + 1))}{t(t + 1)} = -\frac{t(t + 2)}{t + 1} = -\left(\frac{t + 2}{t + 1}\right).$$

- Plot the average velocity $V(t)$ for the time period $[-\frac{1}{2}, 1] - \{0\}$. The plot is part of a rectangular hyperbola, steadily increasing from $[-\frac{1}{2}, -3]$ to the point $[1, -\frac{3}{2}]$ with a point omitted at $[0, -2]$. The curve is concave down.

- What are the values of $V(t)$, when $t = 1$, $t = 0.1$ and $t = 0.01$?

$$V(1) = -1.50, \quad V(0.1) = -\frac{21}{11} = -1.91, \quad V(0.01) = -\frac{201}{101} = -1.99.$$ 

- Find, by taking a suitable limit, the instantaneous velocity of the particle at time $t = 0$ and hence determine the equation of the tangent line to the graph of the position against time, at time $t = 0$.

The required instantaneous velocity $v(0)$ is given by the limit of the average velocity $V(t)$:

$$v(0) = \lim_{t\to 0} V(t) = \lim_{t\to 0} \left( -\left(\frac{t + 2}{t + 1}\right) \right) = -2.$$ 

The tangent line at time $t = 0$ has slope $v(0) = -2$ and passes through the point $[0, 1]$ so has the equation $x - 1 = -2(t - 0)$, or $x = -2t + 1$. 

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