Honors Calculus Quiz 1 Solutions 8/28/3

Question 1

For the following real-valued functions determine their ranges and decide whether or not they are one-to-one and onto their ranges (note that once you have determined the range, the function is automatically onto its range). If a function is invertible, find a formula for its inverse. Illustrate your results graphically.

- \( y = x^2 - 4x + 3 \) for the domain \( \mathbb{R} \).

Let \( f(x) = x^2 - 4x + 3 = (x - 3)(x - 1) \), defined for \( x \in \mathbb{R} \).

We first naively find the inverse, which involves solving the equation \( y = x^2 - 4x + 3 \) for \( x \), in terms of \( y \), with \( y \) regarded as given.

We have \( y + 1 = x^2 - 4x + 4 = (x - 2)^2 \), so \( x - 2 = \pm \sqrt{y + 1} \) and \( x = 2 \pm \sqrt{y + 1} \). Since this solution is ambiguous, we do not expect the inverse to exist, unless, as in the second part of this question, \( x \) is restricted to obey \( x \geq 2 \) which leaves only the solution \( x = 2 + \sqrt{y + 1} \).

We now make these arguments much more precise. First consider the range \( f(\mathbb{R}) \) of \( f \).

- If \( y = f(x) \), then we have \( y + 1 = (x - 2)^2 \geq 0 \), so \( y \geq -1 \), so \( f(\mathbb{R}) \subset [-1, \infty) \).
- Next, given \( y \geq -1 \), put \( x = 2 + \sqrt{y + 1} \), which is well-defined, since \( y + 1 \geq 0 \). Then we have:
  \[
  f(x) = f(2 + \sqrt{y + 1}) = (2 + \sqrt{y + 1} - 2)^2 - 1 = (\sqrt{y + 1})^2 - 1 = y + 1 - 1 = y,
  \]
  so \( y \) is an output for \( f \), so \( [-1, \infty) \subset f(\mathbb{R}) \).
- Putting these results together, we have \( f(\mathbb{R}) \subset [-1, \infty) \subset f(\mathbb{R}) \), so \( f(\mathbb{R}) = [-1, \infty) \), so the function has range the infinite interval \( [-1, \infty) \).

So we have shown that regarded as a map \( f : \mathbb{R} \to [-1, \infty) \), \( f \) is well-defined and onto. But it is certainly not one-to-one: for example we have \( f(3) = f(1) = 0 \), or \( f(4) = f(0) = 3 \), so the horizontal line test fails. So the function has no inverse.
• \( y = x^2 - 4x + 3 \) for the domain \([2, \infty)\).

Let \( g(x) = x^2 - 4x + 3 \), defined for \( x \in [2, \infty) \).

Note that \( g(x) = f(x) \), for any \( x \in [2, \infty) \).

For this new domain, the discussion proceeds almost the same as above: clearly the range, \( g([2, \infty)) \), is a subset of the range found above, the interval \([-1, \infty)\), since there are fewer inputs for the same formula.

Now let \( y \in [-1, \infty) \), and as before, put \( h(y) = 2 + \sqrt{y+1} \), so \( h(y) \) is well-defined.

Then \( h(y) \geq 2 \), for any \( y \in [-1, \infty) \), since \( \sqrt{y+1} \geq 0 \).

So \( h \) gives a well-defined map from \( \mathbb{B} = [-1, \infty) \) to \( \mathbb{A} = [2, \infty) \).

Since \( g : \mathbb{A} \to \mathbb{B} \), the compositions \( g \circ h \) and \( h \circ g \) are both well-defined.

So now we compute these compositions:

- For any \( y \in \mathbb{B} \), we have:

  \[
  g(h(y)) = g(2 + \sqrt{y+1}) = (2 + \sqrt{y+1} - 2)^2 - 1
  \]

  \[
  = (\sqrt{y+1})^2 - 1 = y + 1 - 1 = y.
  \]

  So \( g \circ h = I_{\mathbb{B}} \).

- For any \( x \in \mathbb{A} \), we have:

  \[
  h(g(x)) = 2 + \sqrt{g(x) + 1} = 2 + \sqrt{x^2 - 4x + 3 + 1}
  \]

  \[
  = 2 + \sqrt{(x-2)^2} = 2 + |x-2| = 2 + (x-2) = x.
  \]

  Note that we used the fact that \( x \geq 2 \) for \( x \in \mathbb{A} \) to simplify the quantity \(|x-2|\) to \( x-2 \).

  So \( h \circ g = I_{\mathbb{A}} \).

Putting these results together, since we have both \( g \circ h = I_{\mathbb{B}} \) and \( h \circ g = I_{\mathbb{A}} \), we deduce that \( g \) and \( h \) are each bijections and each is each other’s inverse.

In particular, we that \( g \) is one-to-one and onto, regarded as a map \( g : \mathbb{A} \to \mathbb{B} \) and we have \( g^{-1} = h \).

Also the range of \( g \) is the domain of \( h \), so is \( \mathbb{B} = [-1, \infty) \).
• $y = \frac{1}{x-1}$ for the domain $\mathbb{R} - \{1\}$.

Here we can be quicker, since it is easy to determine the inverse function and the range: naively if $y = \frac{1}{x-1}$, then $\frac{1}{y} = x - 1$ and $x = 1 + \frac{1}{y}$ should be the inverse function. It is only defined for $y \neq 0$, so the range of the given function ought to avoid 0. We now make this precise.

Put $A = \mathbb{R} - \{1\}$, $B = \mathbb{R} - \{0\}$ and $f(x) = \frac{1}{x-1}$ defined for $x \in A$. Also put $g(y) = 1 + \frac{1}{y}$, defined for any $y \in B$.

- It is clear that for any $x \in A$, $x - 1 \neq 0$, so $f(x)$ is well-defined and is non-zero, since the reciprocal of a real number is never zero. So $f(x) \in B$, for any $x \in A$. So $f : A \to B$ is a well-defined map.

- The quantity $g(y)$ is well-defined, for any $y \in B$. Also, if $g(y) = 1$, then $1 + \frac{1}{y} = 1$, so $\frac{1}{y} = 0$, which is impossible, so $g(y)$ is never 1, so $g(y) \in A$. So $g : B \to A$ is a well-defined map.

Since $f : A \to B$ and $g : B \to A$ are well-defined maps, the compositions $f \circ g$ and $g \circ f$ are both well-defined.

So now we compute these compositions:

- For any $y \in B$, we have:
  $$(f \circ g)(y) = f(g(y)) = \frac{1}{g(y) - 1} = \frac{1}{\frac{1}{y} - 1} = \frac{1}{(\frac{1}{y})} = y.$$  
  So $f \circ g = I_B$.

- For any $x \in A$, we have:
  $$(g \circ f)(x) = g(f(x)) = 1 + \frac{1}{f(x)} = 1 + \frac{1}{\frac{1}{(x-1)}} = 1 + (x - 1) = x.$$  
  So $g \circ f = I_A$.

Putting these results together, since we have both $f \circ g = I_B$ and $g \circ f = I_A$, we deduce that $f$ and $g$ are each bijections and each is each other’s inverse. In particular, we have that $f$ is one-to-one and onto, regarded as a map $f : A \to B$ and we have $f^{-1} = g$.

Also the range of $f$ is the domain of $g$, so is $B = \mathbb{R} - \{0\}$. 

Question 2

Consider the iteration (multiple composition of the function with itself) of the function $y = \cos(x)$ on the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Pick three generic starting points and determine what happens to these points after five iterations.

What can you conclude about the eventual behavior of the iteration? Illustrate your results graphically.

We pick $x = 0.5$, $x = 1$ and $x = 1.5$ as our starting points (note these are reasonably generic values on the natural scale of trigonometric functions, which is $\pi$; also we may as well start with $x \geq 0$, since the graph of $\cos(x)$ is symmetrical about the $y$-axis) and compute five iterations, with $f(x) = \cos(x)$, defined for any real $x$:

\[
\begin{array}{cccccc}
  x & f(x) & f(f(x)) & f(f(f(x))) & f(f(f(f(x)))) & f(f(f(f(f(x)))))) \\
 0.5 & 0.878 & 0.639 & 0.803 & 0.695 & 0.719 \\
 1 & 0.540 & 0.858 & 0.654 & 0.793 & 0.701 \\
 1.5 & 0.071 & 0.997 & 0.542 & 0.856 & 0.655 \\
\end{array}
\]

There seems to be a tendency to cluster around the value of about 0.7. This would then be a fixed point of the iteration, so would obey the equation $\cos(x) = x$.

To test this, we extend to twelve iterations, obtaining the following output vectors:

\[
\begin{array}{cccccccccccc}
  x & f(x) & f(f(x)) & f(f(f(x))) & f(f(f(f(x)))) & f(f(f(f(f(x)))))) & f(f(f(f(f(f(x)))))) \\
 0.5 & 0.878 & 0.639 & 0.803 & 0.695 & 0.719 & 0.752 & 0.730 & 0.745 & 0.735 & 0.742 & 0.737 \\
 1 & 0.540 & 0.858 & 0.654 & 0.793 & 0.701 & 0.764 & 0.722 & 0.750 & 0.731 & 0.744 & 0.736 & 0.741 \\
 1.5 & 0.071 & 0.997 & 0.542 & 0.856 & 0.655 & 0.793 & 0.702 & 0.764 & 0.722 & 0.750 & 0.732 & 0.744 \\
\end{array}
\]

There is a definite feeling that we are approaching a fixed point. Indeed, numerically solving the equation $\cos(x) = x$ gives the approximate solution $x = 0.739085133$, so the data indicate that we oscillate alternately above and below this critical value, getting closer to it as the iteration proceeds.

We can refer to the accompanying Maple file to see the graphics.