Notes for Monday 25th August

Sets

A set is a collection of primitive entities called elements. If we know what the elements are we can list them. For example the set \( S \) containing the numbers 1 through 10 can be written:

\[
S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}
\]

Some important sets have more or less standard names:

- \( \phi \), the empty set: no elements.
- \( \mathbb{N} \): the set of all positive integers (also called natural numbers).
  \[
  \mathbb{N} = \{1, 2, 3, \ldots, 10^{10^{10}}, 10^{10^{10}} + 1, \ldots\}
  \]
- \( \mathbb{Z} \): the set of all integers.
  \[
  \mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}
  \]
- \( \mathbb{Q} \): the set of all rationals.
  \[
  \mathbb{Q} = 0, 1, -1, 2, -2, \frac{1}{2}, -\frac{1}{2}, 3, -3, \frac{1}{3}, -\frac{1}{3}, 4, -4, \frac{3}{2}, -\frac{3}{2}, \frac{2}{3}, -\frac{2}{3}, \frac{1}{4}, -\frac{1}{4}, \ldots
  \]
- \( \mathbb{R} \): the set of all real numbers.
- \( \mathbb{R}^+ \): the set of all positive reals.
- \( \mathbb{C} \): the set of all complex numbers.

If \( A \) is a set and if \( x \) is an element of \( A \), we write \( x \in A \), or \( A \ni x \).
If \( y \) is not in \( A \), we write \( y \notin A \).
Often a set is given as a subset of some other set. We write \( B \subset A \) if every element of \( B \) lies in the set \( A \) and then \( B \) is called a subset of \( A \).
So for example \( \mathbb{N} \subset \mathbb{Z} \) and \( \mathbb{R}^+ \subset \mathbb{R} \).
\( x \in A \) if and only if \( \{x\} \subset A \).
Subsets are often delineated by properties obeyed by their elements:

- The set $E$ of all even integers:
  \[ E = \{ n \in \mathbb{Z} : n = 2k, \text{ for some } k \in \mathbb{Z} \} \subset \mathbb{Z} \]

- The closed real interval $[a, b]$ (where $a \in \mathbb{R}$ and $b \in \mathbb{R}$ are given):
  \[ [a, b] = \{ x \in \mathbb{R} : a \leq x \leq b \} \subset \mathbb{R} \]

Two sets are equal if and only if they have the same elements if and only if each set is a subset of the other.

Sometimes we consider families of sets, indexed by another set. This is typically written as follows: $\{S_i; i \in I\}$. Here $I$ is the indexing set.

For example we might define $S_j$ for each $j \in \mathbb{N}$ by $S_j = \{ n \in \mathbb{N} : j \text{ divides } n \}$.

The union of a family is the collection of all elements that belong to at least one member of the family:

\[ \bigcup_{i \in I} S_i = \{ x : x \in S_i, \text{ for some } i \in I \} \]

The intersection of a family is the collection of all elements that belong to every member of the family:

\[ \bigcap_{i \in I} S_i = \{ x : x \in S_i, \text{ for all } i \in I \} \]

If we have only a few sets in the family, for example, three sets, $A$, $B$ and $C$, then we write their union as $A \cup B \cup C$ and their intersection as $A \cap B \cap C$.

The Cartesian product of a pair of sets $A$ and $B$, denoted $A \times B$ is the set of all ordered pairs $(a, b)$, with $a \in A$ and $b \in B$.

Here the set $A$ is thought of as lying along the "horizontal", or $x$-axis and the set $B$ along the "vertical", or $y$-axis.

The Cartesian product of a list of $n$ sets $A_i, i = 1, 2, 3, \ldots, n$, for $n \in \mathbb{N}$ is the set of all ordered $n$-tuples, $(a_1, a_2, a_3, \ldots, a_n)$, with $a_i \in A_i$, for each $i$.

If all the sets in a product are the same we write $A^n$ for the $n$-fold Cartesian product of $A$ with itself (if $n = 1$, we put $A^1 = A$).

For example $\mathbb{R}^3$ is the set of all ordered triples $(x, y, z)$ with $x$, $y$ and $z$ in $\mathbb{R}$.
Maps/functions

A map or function \( f \) is a procedure for taking an input and associating to that input a unique output. Using sets, we have an input set \( A \) and if \( x \in A \), we get an output in some other set \( B \). Then \( A \) is called the domain of \( f \).

We may write this in many different ways:

- \( y = f(x) \): here it is understood that \( x \) lies in \( A \) and then \( y \) in \( B \) is calculated according to the rule \( f \).
- \( x \rightarrow f(x) \): here it is understood that \( x \) lies in \( A \) and then \( f(x) \) in \( B \) is the output.
- Sometimes we sketch the sets \( A \) and \( B \) on a piece of paper and then the function is depicted as a collection of arrows, where each arrow joins an element of \( A \) to its correct output in \( B \).
- \( f : A \rightarrow B \): here we are thinking of the totality of \( f \): it is taking stuff in \( A \) and transforming it systematically into corresponding stuff in \( B \).
- \( f \subset A \times B \): here we collect together the set of all pairs of the form (input, output) associated to the procedure \( f \).

The last approach gives a more formal definition of a function: it is also called the graph of \( f \).

So formally a function \( f \) from \( A \) to \( B \) is a subset \( f \) of \( A \times B \) obeying the following rules:

- If \((x, y) \in f \) and \((x, y') \in f \), then \( y = y' \).
  This says that \( f \) is single-valued, or that there is a unique output given the input. This is also called the vertical line rule.
- If \( x \in A \), then there exists a \( y \in B \), such that \((x, y) \in f \).
  This says that the domain of \( f \) is all of the set \( A \), not just some part of it.

These two rules can be combined into a single rule:

- Given \( x \in A \), there is a unique \( y \in B \), such that \((x, y) \in f \).
Examples:

- The squaring function $f_1 = \{(x, x^2) : x \in \mathbb{R}\} \subset \mathbb{R} \times \mathbb{R}$.
- The squaring function $f_2 = \{(x, x^2) : x \in \mathbb{R}\} \subset \mathbb{R} \times (\mathbb{R}^+ \cup \{0\})$.
- The square root function $f_3 = \{(x, \sqrt{x}) : x \in \mathbb{R}^+ \cup \{0\}\} \subset (\mathbb{R}^+ \cup \{0\})^2$.

Note that we do not have to get all of $\mathbb{B}$ for the output of a function $f : \mathbb{A} \to \mathbb{B}$: indeed for the function $f_1$ we do not get all of $\mathbb{B}$, but for $f_2$ and $f_3$, we do.

Two functions are equal if they have the same domains and always give the same output for the same input: so if they are equal as sets of pairs.

So for example $f_1$ and $f_2$ are equal functions even though they are defined slightly differently, with different $\mathbb{B}$’s.

If $f : \mathbb{A} \to \mathbb{B}$ and if $\mathbb{C} \subset \mathbb{A}$, then $f(\mathbb{C}) \subset \mathbb{B}$, called the image of $\mathbb{C}$ under $f$, is the set of all $y \in \mathbb{B}$ such that $y = f(x)$, for some $x \in \mathbb{C}$.

If $\mathbb{D} \subset \mathbb{B}$, then $f^{-1}(\mathbb{D}) \subset \mathbb{A}$, called the inverse image of $\mathbb{D}$ under $f$, is the set of all $x \in \mathbb{A}$ such that $f(x) \in \mathbb{D}$.

The range of $f$ is $f(\mathbb{A})$, so is the set of all outputs for $f$:

$$f(\mathbb{A}) = \{ y : (x, y) \in f, \text{ for some } x \in \mathbb{A} \}.$$ 

For example for the map $f_1$, we have:

- $f_1(\mathbb{R}) = \mathbb{R}^+ \cup \{0\}$.
- $f_1((-3, 4)) = [0, 16]$.
- $f_1^{-1}([0, 16]) = (-4, 4)$. 

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Three important properties that a function \( f : \mathbb{A} \rightarrow \mathbb{B} \) might have:

- \( f \) is one-to-one, or injective, or a monomorphism, if and only if:
  - Different inputs lead to different outputs.
  - The equation \( f(x) = f(x') \) (for \( x \in \mathbb{A} \) and \( x' \in \mathbb{A} \)) has only the solution \( x = x' \).
  - The equation \( y = f(x) \) with \( y \in \mathbb{B} \) has at most one solution \( x \in \mathbb{A} \).
  - The arrows of \( f \) never meet.
  - If \((x, y) \in f \) and \((x', y) \in f \) then \( x' = x \).
  - A horizontal line intersects the graph of \( f \) at at most one point.
    This is sometimes called the horizontal line rule.

- \( f \) is onto, or surjective, or an epimorphism, if and only if:
  - The outputs use up all of \( \mathbb{B} \).
  - The arrows of \( f \) cover all \( \mathbb{B} \): every \( y \in \mathbb{B} \) is at the end of some arrow.
  - If \( y \in \mathbb{B} \) then \((x, y) \in f \), for some \( x \in \mathbb{A} \).
  - Given any \( y \in \mathbb{B} \), the equation \( y = f(x) \) has at least one solution with \( x \in \mathbb{A} \).

- \( f \) is one-to-one and onto, or bijective, or an isomorphism, or invertible if and only if:
  - The arrows of \( f \) never meet and cover all \( \mathbb{B} \).
  - Reversing the arrows of \( f \) gives a function from \( \mathbb{B} \) to \( \mathbb{A} \).
  - If \( y \in \mathbb{B} \) then \((x, y) \in f \), for some unique \( x \in \mathbb{A} \).
  - Given any \( y \in \mathbb{B} \), the equation \( y = f(x) \) has a unique solution with \( x \in \mathbb{A} \).
  - The set \( f^{-1} = \{(y, x) : (x, y) \in f \} \subset \mathbb{B} \times \mathbb{A} \) is a function.

The set \( f^{-1} \) is then called the inverse function of \( f \) and if \( y = f(x) \), we have \( x = f^{-1}(y) \).
Also we have the relations \( f(f^{-1}(y)) = y \) for all \( y \in \mathbb{B} \) and \( f^{-1}(f(x)) = x \), for all \( x \in \mathbb{A} \).
Composition

Consider two functions \( f : A \rightarrow B \) and \( g : C \rightarrow D \), where the range of \( f \) lies in the domain of \( g \): \( f(A) \subset C \).

Then we can splice the functions together: with input \( x \in A \), we first get the output for \( f \), \( f(x) \in C \) and then use this as the input for \( g \). The resulting output \( g(f(x)) \) lies in \( D \).

The formula \( x \mapsto (g \circ f)(x) = g(f(x)) \) gives a well-defined map from \( A \) to \( D \), called the composition \( g \circ f \) of \( f \) and \( g \).

In set terms, we have:

\[
g \circ f = \{(x, z) \in A \times D : (x, y) \in f \text{ and } (y, z) \in g, \text{ for some } y \in C\}.
\]

Pictorially, the composition is understood as follows: starting at \( x \), we follow the \( f \) arrow to \( f(x) \). Then we join on the \( g \) arrow starting at \( f(x) \) ending in \( D \) at the element \( g(f(x)) \).

From this point of view, composition amounts to joining arrows together, so is clearly associative: if the various compositions are well defined, then we have:

\[
h \circ (g \circ f) = (h \circ g) \circ f.
\]

Proof: acting on \( x \) in the domain of \( f \), both sides evaluate to the quantity \( h(g(f(x))) \). So we do not need to use parentheses and can write just \( h \circ g \circ f \).

Example, where all maps go from \( \mathbb{R} \) to itself:

\[
f : x \rightarrow x^2,
\]

\[
g : x \rightarrow \sin(x),
\]

\[
h : x \rightarrow \frac{1}{x + 2},
\]

\[
g \circ f : x \rightarrow \sin(x^2),
\]

\[
h \circ g : x \rightarrow \frac{1}{\sin(x) + 2},
\]

\[
h \circ g \circ f : x \rightarrow \frac{1}{\sin(x^2) + 2}.
\]
Note that it is sometimes convenient to roughen the idea of composition, by dropping the condition that \( f(A) \subset C \): the weakened composition is then given as follows:

\[
g \circ f = \{(x, z) \in A \times D : (x, y) \in f \text{ and } (y, z) \in g, \text{ for some } y \in C \cap f(A)\}.
\]

This amounts to splicing arrows whenever possible, or equivalently to using the formula \( g(f(x)) \) whenever it make sense. The domain of the composition is then \( f^{-1}(C) \).

For example, if \( f(x) = x^2 \) defined for any real \( x \) and \( g(x) = \sqrt{1 - x} \), defined for any real \( x \leq 1 \), then \( (g \circ f)(x) = \sqrt{1 - x^2} \), defined for \(-1 \leq x \leq 1\).

For any set \( S \), we define the identity map \( I_S = \{(x, x) \in S^2\} \): so \( I_S \) maps each \( x \in S \) to itself. The composition of an identity map \( I_A \) with any map \( f : A \to B \) or with \( g : B \to A \) leaves each of the latter maps unchanged:

\[
f \circ I_A = f, \quad I_A \circ g = g.
\]

Using the language of composition, we say that \( f : A \to B \) and \( g : B \to A \) are inverses of each other iff:

\[
f \circ g = I_B \text{ and } g \circ f = I_A.
\]

Then we have \( f^{-1} = g \) and \( g^{-1} = f \).

For example, the following maps are inverses of each other:

\[
f = \{(x, x^3) \in \mathbb{R}^2 : x \in \mathbb{R}\}
\]

\[
g = \{(x, \sqrt[3]{x}) \in \mathbb{R}^2 : x \in \mathbb{R}\}.
\]

Proof: note that we have, for any real \( x \), the following compositions:

\[
(f \circ g)(x) = f(g(x)) = f(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x,
\]

\[
(g \circ f)(x) = g(f(x)) = g(x^3) = \sqrt[3]{x^3} = x.
\]
Next a slightly more complicated example, where we first find the inverse intuitively and then make it precise:

- Find the inverse of the following function of \( x \in \mathbb{R} \):
  \[
y = f(x) = \frac{2x - 1}{1 - x}.
  \]

Note that if the domain is not specified, we by default understand that the domain is the largest possible set on which the given formula makes sense.

So here the understood domain of \( f \) is \( \mathbb{A} = \{ x \in \mathbb{R} : x \neq 1 \} \).

- To solve this problem, we first naively solve for \( x \) given \( y \), as follows:
  \[
y = \frac{2x - 1}{1 - x},
  \]
  \[
(1 - x)y = 2x - 1,
  \]
  \[
y - xy = 2x - 1.
  \]

Now we isolate all the terms containing \( x \):
  \[
xy + 2x = 1 + y,
  \]
  \[
x(y + 2) = 1 + y,
  \]
  \[
x = \frac{y + 1}{y + 2}.
  \]

In the last step, we have to assume that \( y + 2 \neq 0 \), otherwise the last step makes no sense.

- So we check now that \( y + 2 \) is never zero, for \( y \) in the range of \( f \), so we need to verify that \( f(x) + 2 \neq 0 \), for any \( x \in \mathbb{A} \):

If \( f(x) + 2 = 0 \), then \( f(x) = -2 \), which gives:
  \[
\frac{2x - 1}{1 - x} = -2,
  \]
  \[
2x - 1 = -2(1 - x) = -2 + 2x,
  \]
  \[
1 = 0.
  \]

This is a contradiction, so \(-2\) is not in the range of \( f \), so \( f \) gives a well-defined map from \( \mathbb{A} = \mathbb{R} - \{1\} \) to the set \( \mathbb{B} = \mathbb{R} - \{-2\} \).
To finish off we now put $g(x) = \frac{x+1}{x+2}$, so $g$ is defined on $\mathbb{B}$ (note this is the formula obtained above by first solving the equation $y = f(x)$ and then interchanging $x$ and $y$). Then we need to check that 1 is not in the range of $g$, so that $f \circ g$ is well-defined:

- The equation $g(x) = 1$ gives:
  \[
  \frac{x + 1}{x + 2} = 1,
  \]
  \[
  x + 1 = x + 2,
  \]
  \[
  0 = 1.
  \]
  This a contradiction so $g : \mathbb{B} \to \mathbb{A}$, as required.

- Now that we have the domains working well, we can compute the compositions:
  \[
  (f \circ g)(x) = f(g(x)) = \frac{2g(x) - 1}{1 - g(x)}
  \]
  \[
  = \frac{2\left(\frac{x+1}{x+2}\right) - 1}{1 - \left(\frac{x+1}{x+2}\right)}
  \]
  \[
  = \frac{2(x+1) - (x+2)}{x+2 - (x+1)} = \frac{2x + 2 - x - 2}{x + 2 - x - 1} = \frac{x}{1} = x.
  \]
  \[
  (g \circ f)(x) = g(f(x)) = \frac{f(x) + 1}{f(x) + 2}
  \]
  \[
  = \frac{\left(\frac{2x-1}{1-x}\right) + 1}{\left(\frac{2x-1}{1-x}\right) + 2}
  \]
  \[
  = \frac{(2x - 1) + (1 - x)}{(2x - 1) + 2(1 - x)} = \frac{2x - 1 + 1 - x}{2x - 1 + 2 - 2x} = \frac{x}{1} = x.
  \]
  The first equation is valid for all $x \in \mathbb{B}$ and the second for all $x \in \mathbb{A}$, so we have the required compositions:
  \[
  f \circ g = I_\mathbb{B}, \quad g \circ f = I_\mathbb{A}.
  \]
  So the map $g$ is the inverse of the map $f$ and we are done.

- As a by product of this calculation, we are able to identify precisely the ranges of these maps: $f$ has range $\mathbb{B}$ and $g$ has range $\mathbb{A}$.
Iterations and discrete dynamical systems

A discrete (deterministic) dynamical system is a set \( \mathbb{A} \) representing the configurations of the system and a map \( f : \mathbb{A} \rightarrow \mathbb{A} \) representing the evolution of the system during one time step.

If we measure the time step by integers, we may write the evolution as:

\[
x_{n+1} = f(x_n).
\]

Here \( x_n \in \mathbb{A} \) is the configuration at time \( n \).

The evolution is then determined (recursively or iteratively), for all positive integer times, given the configuration at time 0.

In the case that \( f \) is invertible, the evolution can be continued back in past times and \( x_n \) is defined for all integer \( n \): \( x_{-n} = f^{-1}(x_{n+1}) \), for any integer \( n \).

Even when \( f \) is not invertible, it is sometimes convenient to consider evolutions defined for all integer \( n \).

Example: the Fibonacci evolution (why is it called this?):

\[
x_{n+1} = 1 + \frac{1}{x_n} = f(x_n), \quad f(x) = \frac{1}{x}.
\]

This evolution is time-symmetric: the time reversal law \( f^{-1} \) is the same as the time step law: \( f = f^{-1} \) (check).

This law exhibits a black hole: certain configurations lead inevitably to disaster in finite time!

These we call trapped configurations: for example \( x_n = 0 \) leads to \( x_{n+1} \) being undefined.

Similarly \( x_n = -1 \) leads to \( x_{n+2} \) being undefined.

Can you determine all the trapped configurations and thereby show how to avoid the black hole? These configurations need to be eliminated from the real line to give the set \( \mathbb{A} \) on which the dynamical system is properly defined for all future times.
Also find out what happens to the non-trapped configurations as time passes.

We see that discrete dynamics amounts to the multiple composition of the dynamical law with itself:

\[ x_{n+1} = f(x_n), \]
\[ x_{n+2} = f(x_{n+1}) = f(f(x_n)) = (f \circ f)(x_n), \]
\[ x_{n+3} = f(x_{n+2}) = f((f \circ f)(x_n)) = (f \circ f \circ f)(x_n). \]

Writing \( f^n \) for the \( n \)-fold composition of \( f \) with itself, we have in general:

\[ x_{k+n} = f^n(x_k), \quad x_n = f^n(x_0). \]

So understanding the behavior of a discrete dynamical system is analyzing the multiple compositions of the evolution map with itself.

Finally to iterate a map \( f : \mathbb{A} \rightarrow \mathbb{A} \), we can adopt either of the following graphical procedures:

- The first uses the auxiliary diagonal \( y = x \).
  - Starting at \( x \in \mathbb{A} \) (on the \( x \)-axis), go vertically up to the point \( (x, y) \), where \( y = f(x) \) on the graph of \( f \).
  - Then go horizontally to the point \( (y, y) \) on the diagonal.
  - Then go vertically to the graph to the point \( (y, z) \) on the graph.
    Since \( z = f(y) \) and \( y = f(x) \), we have \( z = (f \circ f)(x) \).

Now iterate, alternating moving horizontally to the diagonal and then vertically to the graph of \( f \). Each time we arrive at the graph of \( f \), one further composition with \( f \) has been achieved.

Often by carrying out this procedure for a suitable variety of initial conditions, we can see what happens to a general evolution.

Important in this analysis are fixed points or cycles: a configuration \( x \) such that for some minimal positive \( n \), called the length of the cycle, we have \( f^n(x) = x \).
Does the Fibonacci evolution have such cycles, of length one (a fixed point: \( f(x) = x \)), or two \( (f(f(x)) = x \neq f(x)) \), etc. and why are these important? Discuss.

Also study the law \( f(x) = \frac{1}{2}(x + \frac{2}{x}) \), from the same viewpoint.

- The second graphical method of composition is a little more subtle and we will state it more generally: the composition of \( y = f(x) \) with \( y = g(x) \).

Instead of plotting the graphs of \( y = f(x) \) and \( y = g(x) \), we plot the graphs of \( y = f(x) \) (which we will call the red curve) and \( x = g(y) \) (so interchanging the axes for the function \( g \), or the reflection in the diagonal of the curve \( y = x \)), which we will call the blue curve).

Then the composition \( g \circ f \) is achieved as follows:

- Starting at a point on the \( x \)-axis, go vertically up to the red curve.
- Then go horizontally to the blue curve.
- Then go down to the \( x \)-axis and the result is the desired composition \( (g \circ f)(x) \).

The proof is simple: we just chase the points as follows:

- The first move goes from \( A = (x, 0) \) up to the curve at the point \( B = (x, f(x)) \).
- The second move goes horizontally from \( B \) to a point \( C \) on the curve \( x = g(y) \) with the same \( y \)-value as the point \( B \), so this point is \( C = (g(y), y) \) where \( y = f(x) \), so \( C = (g(f(x)), f(x)) \).
- The third move drops down to the \( x \)-axis arriving at the point \( D \) with the same \( x \)-co-ordinate as \( C \), so to the the desired destination at the point \( D = (g(f(x)), 0) = ((g \circ f)(x), 0) \), as required.

More generally if we now alternate moving vertically to the red curve and horizontally to the blue curve, we build up alternate compositions \( x \rightarrow f(x) \rightarrow (g \circ f)(x) \rightarrow (f \circ g \circ f)(x) \rightarrow (g \circ f \circ g \circ f)(x) \rightarrow \ldots \) where we terminate either with a horizontal move from the red curve to the \( y \)-axis (for a composition such as \( f \) or \( f \circ g \circ f \)) or a vertical move from the blue curve to the \( x \)-axis (for a composition such as \( g \circ f \circ g \circ f \)).
So now to compose $f$ with itself we just use the special case where $g(x) = f(x)$, so the red curve is $y = f(x)$ and the blue curve is the reflected curve $x = f(y)$.

More generally we can compose arbitrary functions just as easily.

For example to do the composition $f_6 \circ f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1$, we first plot $C_1 : y = f_1(x)$, $C_3 : y = f_3(x)$ and $C_5 : y = f_5(x)$ together with the reflected curves, $C_2 : x = f_2(y)$, $C_4 : x = f_4(y)$ and $C_6 : x = f_6(y)$.

Then we move from the $x$-axis vertically to $C_1$, horizontally to $C_2$, vertically to $C_3$ horizontally to $C_4$, vertically to $C_5$ and horizontally to $C_6$, finally dropping down to the $x$-axis. For an odd number of compositions, we end by going horizontally to the $y$-axis.

We can even do random dynamics this way, where at each time step we randomly choose from a list of evolutions.

For this we plot each law $y = f_1(x)$, $y = f_2(x)$, etc. as red curves and each reflection $x = f_1(x)$, $x = f_2(x)$, etc., as blue curves.

Then starting on the $x$-axis we go up to one of the red curves, go horizontally to a blue curve, vertically to a red curve, etc., ending with either a horizontal move from a red curve to the $y$-axis, or a vertical move from a blue curve to the $x$-axis.