Honors Calculus Exam 1 Solutions 9/26/3

Question 1

Given that \( f(1) = 4 \) and \( f'(1) = 2 \), \( g(4) = 1 \) and \( g'(4) = 3 \), evaluate the following derivatives:

- \( \frac{d}{dt} (g(4t)e^{f(t)})|_{t=1} \).

We have by the product and chain rules and the rule \((e^u)' = e^uu'\):

\[
\frac{d}{dt} (g(4t)e^{f(t)}) = (\frac{d}{dt} (g(4t)))e^{f(t)} + g(4t) \frac{d}{dt} e^{f(t)}
\]

\[
= 4g'(4t)e^{f(t)} + g(4t)e^{f(t)}f'(t) = e^{f(t)}(g(4t)f'(t) + 4g'(4t)).
\]

Putting \( t = 1 \) in this formula gives:

\[
\frac{d}{dt} (g(4t)e^{f(t)})|_{t=1} = e^{f(1)}(g(4)f'(1) + 4g'(4)) = e^4(1(2) + 4(3)) = 14e^4 = 764.3741004.
\]

- \( \frac{d}{dt} \ln(f(g(t)))|_{t=1} \).

By the chain rule, and the rule \((\ln(u))' = \frac{u'}{u}\), we have:

\[
\frac{d}{dt} \ln(f(g(t)))|_{t=1} = \frac{1}{f(g(t))} \frac{d}{dt} f(g(t))|_{t=1}
\]

\[
= \frac{1}{f(g(1))} f'(g(1))g'(4) = \frac{1}{f(1)} f'(1)(3) = \frac{1}{4}(2)(3) = \frac{3}{2}.
\]

- \( \frac{d}{dt} \arctan(g(f(t)))|_{t=1} \).

By the chain rule and the rule \((\arctan(u))' = \frac{u'}{1+u^2}\), we have:

\[
\frac{d}{dt} \arctan(g(f(t)))|_{t=1} = \frac{1}{1 + (g(f(t))^2)} \frac{d}{dt} g(f(t))|_{t=1}
\]

\[
= \frac{1}{(1 + g(f(1))^2)} g'(f(1))f'(1) = \frac{1}{(1 + g(f(1))^2)} g'(f(1))f'(1)
\]

\[
= \frac{1}{(1 + (g(4))^2)} g'(4)(2) = \frac{1}{(1 + 1^2)}(3)(2) = 3.
\]
Question 2

- Find the equations of the tangent lines to the curves \( y = \tan(x) \) and \( y = 2 \sin(2x) \) at their point of intersection in the interval \( 0 < x < \frac{\pi}{2} \).

The double angle formula: \( \sin(2x) = 2 \sin(x) \cos(x) \) is useful here.

The two curves meet where both equations hold at once, so where:

\[
\tan(x) = 2 \sin(2x), \quad \frac{\sin(x)}{\cos(x)} = 2(2 \sin(x) \cos(x)), \quad \sin(x) = 4 \sin(x) \cos^2(x),
\]

\[
0 = \sin(x)(4 \cos^2(x) - 1) = \sin(x)(2 \cos(x) - 1)(2 \cos(x) + 1),
\]

\[
\sin(x) = 0, \quad \text{or} \quad \cos(x) = \frac{1}{2}, \quad \text{or} \quad \cos(x) = -\frac{1}{2},
\]

\[
x = k\frac{\pi}{3}, \quad k \in \mathbb{Z}.
\]

In the given interval, the only solution is \( x = \frac{\pi}{3} \) and then the \( y \)-coordinate is \( y = \tan(\frac{\pi}{3}) = \sqrt{3} \). So the curves meet at the point \( (\frac{\pi}{3}, \sqrt{3}) \).

The tangent to the curve \( y = \tan(x) \) has slope:

\[
y' = \sec^2(x) = 1 + \tan^2(x) = 1 + (\sqrt{3})^2 = 1 + 3 = 4 \quad \text{at} \quad x = \frac{\pi}{3}.
\]

So the tangent line has the equation:

\[
y - \sqrt{3} = 4(x - \frac{\pi}{3}), \quad \text{or} \quad y - 4x = \sqrt{3} - \frac{4\pi}{3}.
\]

The tangent to the curve \( y = 2 \sin(2x) \) has slope:

\[
y' = 4 \cos(2x) = 4 \cos(2(\frac{\pi}{3})) = 4(-\frac{1}{2}) = -2 \quad \text{at} \quad x = \frac{\pi}{3}.
\]

So the tangent line has the equation:

\[
y - \sqrt{3} = (-2)(x - \frac{\pi}{3}), \quad \text{or} \quad y + 2x = \sqrt{3} + \frac{2\pi}{3}.
\]

- Also find the angle between these tangent lines.

If the lines have slopes \( m_1 \) and \( m_2 \) and the angle between them is \( \theta \), then we have: \( \tan(\theta) = \frac{m_1 - m_2}{1 + m_1m_2} \).

Putting \( m_1 = -2 \) and \( m_2 = 4 \), we get \( \tan(\theta) = \frac{-2 - 4}{1 + (-2)4} = \frac{-6}{7} \), giving:

\[
\theta = \arctan\left(\frac{6}{7}\right) = 0.7086262721 \text{ radians} = 40.60129464 \text{ degrees}.
\]

- Also sketch the curves and the tangent lines.

See the Maple graphics for this.
Question 3

By relating each of the following quantities to the derivative of a suitable function, compute the following limits, or explain why the given limit does not exist.
In each case give the function whose derivative is being computed. Alternatively, compute the limit directly.

- The first limit:

  \[
  \lim_{t \to 1} \left( \frac{t \ln(t) - e^{t-1} + \cos(2(t-1))}{t - 1} \right)
  \]

  Put \( f(t) = t \ln(t) - e^{t-1} + \cos(2(t-1)) \).

  Then \( f(1) = 1 \ln(1) - e^{0} + \cos(0) = 0 \), so we have:

  \[
  \lim_{t \to 1} \left( \frac{t \ln(t) - e^{t-1} + \cos(2(t-1))}{t - 1} \right) = \lim_{t \to 1} \left( \frac{f(t) - f(1)}{t - 1} \right) = f'(1)
  \]

  \[
  = \left[ \frac{d}{dt} (t \ln(t) - e^{t-1} + \cos(2(t-1))) \right]_{t=1}
  \]

  \[
  = [1 \ln(t) + e^{t-1}(1) - 2 \sin(2(t-1))]_{t=1}
  \]

  \[
  = \ln(1) + 1 - e^{0} - 2 \sin(0) = 0.
  \]

- The second limit:

  \[
  \lim_{t \to 0} \left( \frac{\sqrt{1 + t} - 1}{t \sqrt{1 + t^2}} \right)
  \]

  We compute this limit directly:

  \[
  \lim_{t \to 0} \left( \frac{\sqrt{1 + t} - 1}{t \sqrt{1 + t^2}} \right) = \frac{1}{\sqrt{1 + t^2}} \lim_{t \to 0} \left( \frac{\sqrt{1 + t} - 1}{t} \right)
  \]

  \[
  = \lim_{t \to 0} \left( \frac{(\sqrt{1 + t} - 1)(\sqrt{1 + t} + 1)}{t(\sqrt{1 + t} + 1)} \right)
  \]

  \[
  = \frac{1}{\sqrt{1 + t^2}} \lim_{t \to 0} \left( \frac{(\sqrt{1 + t})^2 - 1}{t} \right)
  \]

  \[
  = \frac{1}{2} \lim_{t \to 0} \left( \frac{1 + t - 1}{t} \right) = \frac{1}{2} \lim_{t \to 0} \left( \frac{t}{t} \right) = \frac{1}{2}.
  \]

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Alternatively, we can first put:

\[ f(t) = \frac{\sqrt{1 + t} - 1}{\sqrt{1 + t^2}}. \]

Then \( f(0) = 0 \), so the given limit is:

\[
\lim_{t \to 0} \left( \frac{\sqrt{1 + t} - 1}{t\sqrt{1 + t^2}} \right) = \lim_{t \to 0} \left( \frac{f(t) - f(0)}{t - 0} \right) = f'(0)
\]

\[
= \left[ \frac{d}{dt} \left( \frac{\sqrt{1 + t} - 1}{\sqrt{1 + t^2}} \right) \right]_{t=0}
\]

\[
= \left[ \frac{1}{1 + t^2} \left( \sqrt{1 + t^2} \left( \frac{1}{2} (1 + t)^{-\frac{1}{2}} \right) - (\sqrt{1 + t} - 1) \left( \frac{1}{2} (2t)(1 + t^2)^{-\frac{1}{2}} \right) \right) \right]_{t=0} = \frac{1}{2}.
\]

- The third limit:

\[
\lim_{t \to 1} \left( \frac{\arcsin(t) - 2 \arctan(t)}{t - 1} \right)
\]

We put:

\[ f(t) = \arcsin(t) - 2 \arctan(t). \]

Then we have \( f(1) = \arcsin(1) - 2 \arctan(1) = \frac{\pi}{2} - 2 \left( \frac{\pi}{4} \right) = 0 \), so the given limit is:

\[
\lim_{t \to 1} \frac{f(t) - f(1)}{t - 1} = f'(1),
\]

provided that the derivative exists.

But \( f'(t) = \frac{1}{\sqrt{1 - t^2}} - \frac{2}{1 + t^2} \), which goes to infinity as \( t \to 1 \) and the derivative of \( \arcsin(t) \) at \( t = 1 \) does not exist:

In fact since \( \sin(\arcsin(t)) = t \), for any \( t \in [-1, 1] \), if the function \( \arcsin(t) \) were differentiable at \( t = 1 \), with derivative \( a \), we would get, by the chain rule:

\[
1 = \left[ \frac{d}{dt} \right]_{t=1} = \left[ \frac{d}{dt} \sin(\arcsin(t)) \right]_{t=1} = [\cos(\arcsin(t))] \frac{d}{dt} \arcsin(t)]_{t=1} = \cos(\frac{\pi}{2})a = 0a = 0.
\]

So \( 1 = 0 \), a contradiction.

So the required limit does not exist.
Question 4

- Find a formula for the inverse function of the following function:

\[ f(x) = \frac{3x - 2}{x - 1}. \]

- Give appropriate domains for the function \( f \) and for its inverse \( f^{-1} \) and verify that their compositions are well-defined.

- Also compute the compositions \( f \circ f^{-1} \) and \( f^{-1} \circ f \) explicitly.

- Find an interval around the point \( x = 2 \), such that for all \( x \) in the interval, we have the inequality \( 3.99 < f(x) < 4.01 \).

- Plot the graphs of the functions \( f \) and \( f^{-1} \) on the same graph.

We first solve the equation \( y = \frac{3x - 2}{x - 1} \) for \( x \) given \( y \):

\[ y = \frac{3x - 2}{x - 1}, \quad y(x - 1) = 3x - 2, \]
\[ yx - y = 3x - 2, \quad yx - 3x = y - 2, \quad x(y - 3) = y - 2, \]
\[ x = \frac{y - 2}{y - 3}. \]

For the last step, we need to assume that \( y \neq 3 \).

Introduce two subsets of \( \mathbb{R} \), \( A = \mathbb{R} - \{1\} \) and \( B = \mathbb{R} - \{3\} \) and two functions:

\[ f : A \to \mathbb{R}, f(x) = \frac{3x - 2}{x - 1}, \text{ for any } x \in A, \]
\[ g : B \to \mathbb{R}, g(x) = \frac{x - 2}{x - 3}, \text{ for any } x \in B. \]

Then \( f \) and \( g \) are well-defined.

- If \( x \in A \) and \( f(x) = 3 \), we get \( \frac{3x - 2}{x - 1} = 3, \quad 3x - 2 = 3(x - 1) = 3x - 3, \quad 1 = 0 \), a contradiction.
  So \( 3 \notin f(A) \), so \( f(A) \subset B \) and \( f \) gives a well-defined map \( f : A \to B \).

- If \( x \in B \) and \( g(x) = 1 \), we get \( \frac{x - 2}{x - 3} = 1, \quad x - 2 = x - 3, \quad 1 = 0 \), a contradiction.
  So \( 1 \notin g(B) \), so \( g(B) \subset A \) and \( g \) gives a well-defined map \( g : B \to A \).
So the compositions $f \circ g : \mathbb{B} \to \mathbb{B}$ and $g \circ f : \mathbb{A} \to \mathbb{A}$ are well-defined. We next compute these compositions explicitly:

- First the composition $f \circ g$.
  For each $x \in \mathbb{B}$ we have:

  \[
  (f \circ g)(x) = f(g(x)) = \frac{3g(x) - 2}{g(x) - 1} = \frac{3\left(\frac{x-2}{x-3}\right) - 2}{\left(\frac{x-2}{x-3}\right) - 1}
  \]

  \[
  = \frac{3(x-2) - 2(x-3)}{x-2 - (x-3)} = \frac{3x - 6 - 2x + 6}{x - 2 - x + 3} = x.
  \]

- Next the composition $g \circ f$.
  For each $x \in \mathbb{A}$ we have:

  \[
  (g \circ f)(x) = g(f(x)) = \frac{f(x) - 2}{f(x) - 3} = \frac{\left(\frac{x-2}{x-1}\right) - 2}{\left(\frac{x-2}{x-1}\right) - 3}
  \]

  \[
  = \frac{3x - 2 - 2(x-1)}{3x - 2 - 3(x-1)} = \frac{3x - 6 - 2x + 2}{3x - 2 - 3x + 3} = x.
  \]

This shows that $f \circ g = \text{Id}_\mathbb{B}$ and $g \circ f = \text{Id}_\mathbb{A}$, so the functions $f$ and $g$ are each other’s inverses.

The domain of $f$ is $\mathbb{A}$. The domain of $f^{-1}$ is $\mathbb{B}$.

To solve the inequality $3.99 < f(x) < 4.01$, we note that the derivatives of $f$ and $g$:

\[
f'(x) = \frac{d}{dx} \left( \frac{3x - 2}{x - 1} \right) = \frac{1}{(x-1)^2} (3(x-1) - (3x - 2)(1)) = -\frac{1}{(x-1)^2} < 0,
\]

\[
g'(x) = \frac{d}{dx} \left( \frac{x - 2}{x - 3} \right) = \frac{1}{(x-3)^2} ((x-3)(1) - (x - 2)(1)) = -\frac{1}{(x-3)^2} < 0.
\]

So $g$ is a decreasing function on the interval $(3.99, 4.01)$, so applying $g$ to the inequality reverses the inequality signs, giving:

\[g(3.99) > g(f(x)) > g(4.01), \quad g(3.99) > x > g(4.01).\]

Conversely if $g(3.99) > x > g(4.01)$, then applying $f$ to this inequality gives $3.99 < f(x) < 4.01$. 

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So the interval \((g(4.01), g(3.99))\) is the set of all solutions of the given inequality.

Evaluating \(g(4.01)\) gives \(\frac{201}{101}\) and evaluating \(g(3.99)\) gives \(\frac{199}{99}\).

So the interval \(J = (\frac{201}{101}, \frac{199}{99})\) solves the given inequality.

Note that \(201 = 2 - \frac{1}{101} < 2\) and \(199 = 2 + \frac{1}{99} > 2\), so \(2 \in J\), as required.

Alternatively, we may solve the inequality directly:

\[
3.99 < f(x) < 4.01,
\]
\[
3.99 < \frac{3x - 2}{x - 1} < 4.01.
\]
\[
3.99 < \frac{3x - 3 + 1}{x - 1} < 4.01,
\]
\[
3.99 < \frac{3}{x - 1} + \frac{1}{x - 1} < 4.01,
\]
\[
3.99 < 3 + \frac{1}{x - 1} < 4.01,
\]
\[
0.99 < \frac{1}{x - 1} < 1.01,
\]
\[
\frac{100}{101} < x - 1 < \frac{100}{99},
\]
\[
\frac{201}{101} < x < \frac{199}{99}.
\]

Every step in this derivation may be reversed, so the solution space is the open interval \(J = (\frac{201}{101}, \frac{199}{99})\), as before.

For the last part, see the accompanying Maple graphics files.
Question 5

- A dangerous radioactive material is to be used in a medical procedure on patient X.

- The half-life of the material is four hours.

- At 9am Monday morning a sealed capsule containing 10 grams of the material is delivered to the facility where the procedure is to be carried out.

- The capsule is to be administered to patient X, when 4 grams of the material remain.
  When will that be?

- The remaining material is flushed out of patient X, when 1 milligram of the material remains.
  When will that be?

Since the half-life is four hours, in units of four hours, the amount of material halves each unit, so the amount $A(t)$ in grams, at time $t$ hours after delivery is given by the formula:

$$A(t) = 10 \left(\frac{1}{2}\right)^{\frac{t}{4}} = 10(2^{-\frac{t}{4}}).$$

The amount becomes 4 grams at time $t$ hours, given by the solution of the equation:

$$4 = 10(2^{-\frac{t}{4}}), \quad 2^{-\frac{t}{4}} = \frac{10}{4} = \frac{5}{2},$$

$$\frac{t}{4} \ln(2) = \ln\left(\frac{5}{2}\right) = \ln(5) - \ln(2),$$

$$t = 4 \frac{\ln(5) - \ln(2)}{\ln(2)} = 4 \frac{\ln(5)}{\ln(2)} - 4 = 5.287712376.$$

Now 5.287712376 hours is 5 hours, 17 minutes and 15.76455 seconds, so the capsule should be administered at about sixteen seconds after 2.17pm Monday afternoon.
Finally, one milligram remains at time $t$ hours, when we have:

\[ 10^{-3} = 10(2^{-\frac{t}{4}}), \quad 2^{\frac{t}{4}} = 10^4, \]

\[ 2^t = (2^{\frac{t}{4}})^4 = (10^4)^4 = 10^{16}, \quad t \ln(2) = 16 \ln(10), \]

\[ t = \frac{16 \ln(10)}{\ln(2)} = 16 \frac{\ln(5) + \ln(2))}{\ln(2)} = 16 + 16 \frac{\ln(5)}{\ln(2)} = 53.15084952. \]

Now 53.15084952 hours is 2 days, 5 hours, 9 minutes and 0.0583 seconds, so the capsule should be flushed out at about three seconds after 2.09pm on the first Wednesday afternoon after the capsule was administered.

Alternatively we can use an exponential formalism, writing:

\[ A(t) = 10e^{-kt}. \]

Then $k$ is determined by the formula:

\[ 5 = 10e^{-4k}, \quad e^{4k} = 2, \]

\[ 4k = \ln(2), \quad k = \frac{1}{4} \ln(2). \]

Then the amount becomes 4 grams, when we have:

\[ 4 = 10e^{-kt}, \]

\[ e^{kt} = \frac{10}{4} = \frac{5}{2}, \]

\[ kt = \ln\left(\frac{5}{2}\right), \]

\[ t = \frac{1}{k} \ln\left(\frac{5}{2}\right) = 4 \frac{\ln(5) - \ln(2)}{\ln(2)} = 4 \frac{\ln(5)}{\ln(2)} - 4. \]

This agrees with our previous result.

For the last part, we want:

\[ 10^{-3} = 10e^{-kt}, \]

\[ e^{kt} = 10^4, \]

\[ kt = \ln(10^4) = 4 \ln(10), \]

\[ t = 4 \frac{\ln(10)}{k} = 16 \frac{\ln(10)}{\ln(2)}. \]

Again, this agrees with our previous result.
Question 6

Consider the ellipse with the following equation in the plane:

\[ x^2 + 2xy + 4y^2 = 108. \]

- Find the four points, where the tangents to the ellipse are either vertical or horizontal.
  
  - First we differentiate implicitly, with respect to \( x \), regarding \( y \) as a function of the variable \( x \):
    \[ 2x + 2y + 2xy' + 8yy' = 0. \]
    
    We want \( y' = 0 \) for horizontal tangents, which gives the equation:
    \[ 2x + 2y = 0, \quad y = -x. \]

    Inserting this into the equation of the ellipse gives the relation:
    \[ x^2 + 2x(-x) + 4(-x)^2 = 108, \quad x^2 - 2x^2 + 4x^2 = 108, \]
    \[ 3x^2 = 108, \quad x^2 = 36, \quad (x, y) = \pm (6, -6). \]

  - Next we differentiate implicitly, with respect to \( y \), regarding \( x \) as a function of the variable \( y \):
    \[ 2xx' + 2yx' + 2x + 8y = 0. \]
    
    We want \( x' = 0 \) for vertical tangents, which gives the equation:
    \[ 2x + 8y = 0, \quad x = -4y. \]

    Inserting this into the equation of the ellipse gives the relation:
    \[ (-4y)^2 + 2(-4y)y + 4y^2 = 108, \quad 16y^2 - 8y^2 + 4y^2 = 108, \]
    \[ 12y^2 = 108, \quad y^2 = 9, \quad (x, y) = \pm (12, -3). \]

    So the four points in question, \( A, B, C \) and \( D \), say, are as follows:
    \[ A = (12, -3), \quad B = (6, -6), \quad C = (-12, 3), \quad D = (-6, 6). \]
• These four points form a box inside the ellipse: find its area.
The four points form a parallelogram inside the ellipse. 
A nice formula for the area of a parallelogram is $|ad - bc|$, where $[a, b]$ 
and $[c, d]$ are vectors connecting pairs of vertices, along the non-parallel 
sides.

- One such vector is the vector from $B$ to $A$, which is:
  $$ \mathbf{BA} = [12 - 6, -3 - (-6)] = [6, 3]. $$
- The second such vector is the vector connecting $B$ to $C$, which is:
  $$ \mathbf{BC} = \mathbf{C} - \mathbf{B} = [-12 - 6, 3 - (-6)] = [-18, 9]. $$

So we may take $[a, b] = [6, 3]$ and $[c, d] = [-18, 9]$, giving the required
area $\Delta$ as:

$$ \Delta = ad - bc = 6(9) - 3(-18) = 54 + 54 = 108. $$

Alternatively, we may use the equation $\Delta = bh$, where $b$ is the base of 
the parallelogram and $h$ its height.

- If we take $BC$ as the base, by Pythagoras, we have:
  $$ b = \sqrt{(-12 - 6)^2 + (3 - (-6))^2} = \sqrt{324 + 81} = \sqrt{405} = 9\sqrt{5}. $$

- To get the height we see that the line $BC$ has slope $m$ given by 
  the formula:
  $$ m = \frac{3 - (-6)}{-12 - 6} = \frac{9}{-18} = -\frac{1}{2}. $$
  Then the line $L$ through $B = (6, -6)$, perpendicular to $BC$, has 
slope $-\frac{1}{m} = 2$, so its equation is:
  $$ y - (-6) = 2(x - 6), \text{ or } y - 2x = -18. $$

- Next, the line $AD$ is parallel to $BC$, has slope $-\frac{1}{2}$ and passes 
  through $A = (12, -3)$, so has the equation:
  $$ y - (-3) = -\frac{1}{2}(x - 12), \text{ or } 2y + x = 6. $$
- The lines \( AD \) and \( L \) meet at the point \( P \), where both equations hold at once, so where:

\[
2y + x = 6, \quad y - 2x = -18, \\
y = 2x - 18, \quad 2(2x - 18) + x = 6, \\
5x = 42, \quad x = \frac{42}{5}, \quad y = 2x - 18 = \frac{1}{5}(84 - 90) = -\frac{6}{5}; \\
P = \frac{1}{5}(42, -6).
\]

- Then the height \( h \) is the distance from \( B \) to \( P \), so is:

\[
h = \sqrt{\left(\frac{42}{5} - 6\right)^2 + \left(-\frac{6}{5} - (-6)\right)^2} = \frac{1}{5}\sqrt{(42 - 30)^2 + (-6 + 30)^2} \\
= \frac{1}{5}\sqrt{12^2 + 24^2} = \frac{12}{5}\sqrt{1^2 + 2^2} = \frac{12}{\sqrt{5}}.
\]

So finally the required area is:

\[
\Delta = bh = 9\sqrt{5}\left(\frac{12}{\sqrt{5}}\right) = 9(12) = 108.
\]

This agrees with our previous calculation.

- The four tangents form a box outside the ellipse: find its area.
  The four tangents form a rectangular box of size \( 24 \times 12 \), of area 288.

- Hence give an estimate of the area of the ellipse.
  If the area of the ellipse is \( S \), we now know that we have:

\[
108 < S < 288.
\]

So the best estimate we can give is the average:

\[
S = \frac{1}{2}(288 + 108) = \frac{396}{2} = 198.
\]

Then the error in this estimate is at most 90.
Using Maple, the exact area is \( 36\pi\sqrt{3} \approx 195.8903314 \), so our crude estimate is correct to about 1.08 percent.
Not bad!

- Also sketch the ellipse and the two boxes.
  See the Maple graphics for this.