Honors Calculus Exam 1 9/26/3

Name                                      Signature

Show your work.
20 points per question.
The best four questions will count.

Question 1

Given that \( f(1) = 4 \) and \( f'(1) = 2 \), \( g(4) = 1 \) and \( g'(4) = 3 \), evaluate the following derivatives:

- \( \frac{d}{dt}(g(4t)e^{f(t)})|_{t=1} \).

- \( \frac{d}{dt} \ln(f(g(t)))|_{t=4} \).

- \( \frac{d}{dt} \arctan(g(f(t)))|_{t=1} \).
Question 2

- Find the equations of the tangent lines to the curves \( y = \tan(x) \) and \( y = 2\sin(2x) \) at their point of intersection in the interval \( 0 < x < \frac{\pi}{2} \).

The double angle formula: \( \sin(2x) = 2\sin(x)\cos(x) \) is useful here.

- Also find the angle between these tangent lines.

- Also sketch the curves and the tangent lines.
Question 3

By relating each of the following quantities to the derivative of a suitable function, compute the following limits, or explain why the given limit does not exist.
In each case give the function whose derivative is being computed.
Alternatively compute the limit directly.

\[
\lim_{t \to 1} \left( \frac{t \ln(t) - e^{t-1} + \cos(2(t-1))}{t - 1} \right)
\]

\[
\lim_{t \to 0} \left( \frac{\sqrt{1 + t} - 1}{t \sqrt{1 + t^2}} \right)
\]

\[
\lim_{t \to 1} \left( \frac{\arcsin(t) - 2 \arctan(t)}{t - 1} \right)
\]
Question 4

• Find a formula for the inverse function of the following function:

\[ f(x) = \frac{3x - 2}{x - 1}. \]

• Give appropriate domains for the function \( f \) and for its inverse \( f^{-1} \) and verify that their compositions are well-defined.

• Also compute the compositions \( f \circ f^{-1} \) and \( f^{-1} \circ f \) explicitly.

• Find an interval around the point \( x = 2 \), such that for all \( x \) in the interval, we have the inequality \( 3.99 < f(x) < 4.01 \).

• Plot the graphs of the functions \( f \) and \( f^{-1} \) on the same graph.
Question 5

- A dangerous radioactive material is to be used in a medical procedure on patient $X$.
- The half-life of the material is four hours.
- At 9am Monday morning a sealed capsule containing 10 grams of the material is delivered to the facility where the procedure is to be carried out.
- The capsule is to be administered to patient $X$, when 4 grams of the material remain. When will that be?
- The remaining material is flushed out of patient $X$, when 1 milligram of the material remains. When will that be?
Question 6

Consider the ellipse with the following equation in the plane:

\[ x^2 + 2xy + 4y^2 = 108. \]

- Find the four points, where the tangents to the ellipse are either vertical or horizontal.

- These four points form a box inside the ellipse: find its area.

- The four tangents form a box outside the ellipse: find its area.

- Hence give an estimate of the area of the ellipse.

- Also sketch the ellipse and the two boxes.