Theoretical Mathematics Quiz 3 Solutions 4/2/14

Question 1

Solve the inequality \( \left| \frac{3x + 4}{2x + 1} \right| < 2 \), writing the solution as a union of open real intervals. We need \( x \neq -\frac{1}{2} \) and

\[
\frac{3x + 4}{2x + 1} < 2,
\]

\[
\frac{3x + 4}{2x + 1} > -2.
\]

For the first inequality, we bring the 2 to the left and put under a common denominator:

\[
\frac{3x + 4}{2x + 1} < 2,
\]

\[
\frac{3x + 4}{2x + 1} - 2 < 0,
\]

\[
\frac{3x + 4 - 2(2x + 1)}{2x + 1} < 0,
\]

\[
\frac{3x + 4 - 4x - 2}{2x + 1} < 0,
\]

\[
\frac{2 - x}{2x + 1} < 0,
\]

\[
\frac{x - 2}{2x + 1} > 0.
\]

In the last step, we multiplied by \(-1\) which changes the direction of the inequality.
Now $\frac{a}{b} > 0$, for real numbers $a$ and $b$, with $b \neq 0$, if and only if $a$ and $b$ have the same sign. So we get either:

- $x - 2 > 0$ and $2x + 1 > 0$, so $x > 2$ and $x > -\frac{1}{2}$, so $x > 2$,

or

- $x - 2 < 0$ and $2x + 1 < 0$, so $x < 2$ and $x < -\frac{1}{2}$, so $x < -\frac{1}{2}$,

So the solution space so far is either $x > 2$ or $x < -\frac{1}{2}$.

For the second inequality, we bring the $-2$ to the left and put under a common denominator:

$$\frac{3x + 4}{2x + 1} > -2, \quad \frac{3x + 4}{2x + 1} + 2 > 0,$$

$$\frac{3x + 4 + 2(2x + 1)}{2x + 1} > 0, \quad \frac{3x + 4 + 4x + 2}{2x + 1} > 0,$$

$$\frac{7x + 6}{2x + 1} > 0,$$

So we get either:

- $7x + 6 > 0$ and $2x + 1 > 0$, so $x > -\frac{7}{6}$ and $x > -\frac{1}{2}$, so $x > -\frac{1}{2}$,

or

- $7x + 6 < 0$ and $2x + 1 < 0$, so $x < -\frac{7}{6}$ and $x < -\frac{1}{2}$, so $x < -\frac{7}{6}$,

So the solution space here is either $x > -\frac{1}{2}$ or $x < -\frac{7}{6}$.

To finish we put our results so far together, so we need:

$$\left( x > 2 \text{ or } x < -\frac{1}{2} \right) \text{ and } \left( x > -\frac{1}{2} \text{ or } x < -\frac{7}{6} \right).$$

- If $x > -\frac{1}{2}$, this reduces to just $x > 2$ and $x > -\frac{1}{2}$, so to $x > 2$.

- If $x < -\frac{1}{2}$, this reduces to just $x < -\frac{7}{6}$ and $x < -\frac{1}{2}$, so to $x < -\frac{7}{6}$.

So the required solution set is the union of two open intervals $(-\infty, -\frac{7}{6}) \cup (2, \infty)$. 

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Question 2

For any \( n \in \mathbb{N} \), put \( x_n = \frac{n^2 + n + 1}{n^2 - n + 1} \).
Let \( \mathcal{X} = \{ x_n : n \in \mathbb{N} \} \).

Prove that the sequence \( \mathcal{X} \) is monotonic and determine, with proof, the limit: \( \lim_{n \to \infty} x_n \). Also determine, with proof, \( \sup(\mathcal{X}) \) and \( \inf(\mathcal{X}) \).

We have:

\[
x_m - x_n = \frac{m^2 + m + 1}{m^2 - m + 1} - \frac{n^2 + n + 1}{n^2 - n + 1}
= \frac{(m^2 + 1 + m)(n^2 + 1 - n) - (m^2 + 1 - m)(n^2 + 1 + n)}{(m^2 - m + 1)(n^2 - n + 1)}
= \frac{(m^2 + 1)(n^2 + 1) + m(n^2 + 1) - n(m^2 + 1) - nm - (m^2 + 1)(n^2 + 1) - m(n^2 + 1) + n(m^2 + 1) - nm)}{(m^2 - m + 1)(n^2 - n + 1)}
= \frac{2m(n^2 + 1) - 2mn - 2n(m^2 + 1) + 2mn - 2m - 2n}{(m^2 - m + 1)(n^2 - n + 1)}
= \frac{2m(n - m) + 2(m - n)}{(m^2 - m + 1)(n^2 - n + 1)}
\]

Now since \( m \geq 1 \) and \( n \geq 1 \), we have \( mn \geq n \) and \( n \geq 1 \), so \( mn \geq 1 \), so \( 1 - mn \leq 0 \). So \( x_m - x_n \) is either zero or has the opposite sign as does \( m - n \). So the sequence \( \mathcal{X} \) is monotonic decreasing.

Next \( x_n > 1 \), for all \( n \in \mathbb{N} \), since the numerator exceeds the denominator of the defining formula for \( x_n \) and both are positive (since \( n^2 - n = n(n - 1) \geq 0 \). Also \( x_n \leq x_1 = 3 \), for all \( n \in \mathbb{N} \), since the sequence \( \mathcal{X} \) is monotonic decreasing. So \( \mathcal{X} \) is bounded and \( \inf(\mathcal{X}) = 3 \). Also since \( \mathcal{X} \) is bounded, \( \sup(\mathcal{X}) \) exists and by the monotone convergence theorem, we have \( \sup(\mathcal{X}) = \lim_{n \to \infty} x_n \).

But we have, given \( \epsilon > 0 \), for \( n \geq 2 \), so \( n^2 - 2n > 0 \):

\[
|x_n - 1| = \left| \frac{n^2 + n + 1}{n^2 - n + 1} - 1 \right| = \left| \frac{n^2 + n + 1 - (n^2 - n + 1)}{n^2 - n + 1} \right|
= \left| \frac{2n}{\frac{1}{2}n^2 + \frac{1}{2}(n^2 - 2n) + 1} \right| \leq \frac{2n}{\frac{1}{2}n^2} = \frac{4}{n} < \epsilon, \text{ provided } n > N(\epsilon) = \max \left( \frac{4}{\epsilon}, 2 \right).
\]

This gives \( \sup(\mathcal{X}) = \lim_{n \to \infty} x_n = 1 \) and we are done.
Question 3

Let \( x_n = \frac{(2n - 1)^2}{n^2} \), for any positive integer \( n \).

* Prove, from first principles, that \( \lim_{n \to \infty} x_n = 4 \).

Given \( \epsilon > 0 \), we have, since \( n \geq 1 \), so \( n^2 > 0 \) and \( 4n - 1 \geq 3 > 0 \), for any \( n \in \mathbb{N} \):

\[
|x_n - 4| = \left| \frac{(2n - 1)^2}{n^2} - 4 \right|
= \left| \frac{(2n - 1)^2 - 4n^2}{n^2} \right|
= \left| \frac{4n^2 - 4n + 1 - 4n^2}{n^2} \right|
= \left| \frac{-4n + 1}{n^2} \right|
= \frac{4n - 1}{n^2} < \frac{4n}{n^2} = \frac{4}{n} < \epsilon, \text{ provided } n > N(\epsilon) = \frac{4}{\epsilon}.

This proves that \( \lim_{n \to \infty} x_n = 4 \), as required.

* Also find \( N \in \mathbb{R} \), such that \( |x_n - 4| < 10^{-6} \), for all integers \( n > N \).

By the above analysis, we take \( \epsilon = 10^{-6} \), so we get \( N(\epsilon) = \frac{4}{10^{-6}} = 4(10^6) \).

So for any \( n > N = 4(10^6) \), we have \( |x_n - 4| < 10^{-6} \) and we are done. In fact, the required inequality actually holds for \( n = 4(10^6) \), since then we have \( 4 - x_n = -\frac{1}{160000000000000} \) and for all higher \( n \in \mathbb{N} \) and this is the best possible, since the inequality fails for \( n = 4(10^6) - 1 \), when we have \( 4 - x_n = 4(10^{-6}) + \frac{299999}{15999992000001000000} \).
Question 4

For each $n \in \mathbb{N}$, let $x_n = \frac{n + 2}{n + 3}$.
Given $\epsilon > 0$, find $N(\epsilon)$ such that for all $n > N(\epsilon)$, we have $|x_n - 1| < \epsilon$.
Hence find, with proof, each of the following limits:

- $A = \lim_{n \to \infty} \left( (-1)^n(1 - x_n) + \frac{3}{x_n^2 + x_n + 1} \right)$
- $B = \lim_{n \to \infty} \left( \sqrt{1 + nx_n} - \sqrt{nx_n} \right)$

Given $\epsilon > 0$, we have, since $n + 3 > n > 0$, for any $n \in \mathbb{N}$

$$|x_n - 1| = \left| \frac{n + 2}{n + 3} - 1 \right| = \left| \frac{n + 2 - (n + 3)}{n + 3} \right| = \left| \frac{-1}{n + 3} \right| = \frac{1}{n + 3} < \frac{1}{n} < \epsilon,$$

provided $n > N(\epsilon) = \frac{1}{\epsilon}$.

This proves that $\lim_{n \to \infty} x_n = 1$.
Then we get: $\lim_{n \to \infty} x_n - 1 = 0$, so by known limit properties, this gives also $\lim_{n \to \infty} |x_n - 1| = 0$.

So, again by known limit properties, since $|(-1)^n(x_n - 1)| = |x_n - 1|$, for each $n \in \mathbb{N}$, we get also $\lim_{n \to \infty} (-1)^n(x_n - 1) = 0$.

So now we have for the limit $A$:

$$A = \lim_{n \to \infty} \left( (-1)^n(1 - x_n) + \frac{3}{x_n^2 + x_n + 1} \right)$$

$$= \lim_{n \to \infty} ((-1)^n(1 - x_n)) + \lim_{n \to \infty} \left( \frac{3}{x_n^2 + x_n + 1} \right)$$

$$= 0 + \frac{3}{\lim_{n \to \infty}(x_n^2 + x_n + 1)} = \frac{3}{1^2 + 1 + 1} = 1.$$
For the limit $B$, we first rationalize:

$$B = \lim_{n \to \infty} (\sqrt{1 + nx_n} - \sqrt{nx_n})$$

$$= \lim_{n \to \infty} \frac{(\sqrt{1 + nx_n} - \sqrt{nx_n}) (\sqrt{1 + nx_n} + \sqrt{nx_n})}{\sqrt{1 + nx_n} + \sqrt{nx_n}}$$

$$= \lim_{n \to \infty} \frac{(\sqrt{1 + nx_n})^2 - (\sqrt{nx_n})^2}{\sqrt{1 + nx_n} + \sqrt{nx_n}}$$

$$= \lim_{n \to \infty} \frac{1 + nx_n - nx_n}{\sqrt{1 + nx_n} + \sqrt{nx_n}}$$

$$= \lim_{n \to \infty} \frac{1}{\sqrt{1 + nx_n} + \sqrt{nx_n}}$$

$$= \lim_{n \to \infty} \frac{1}{\sqrt{n} \left(\sqrt{\frac{1}{n} + x_n} + \sqrt{x_n}\right)}$$

But since $x_n \to 1$ and $\frac{1}{n} \to 0$, as $n \to \infty$, we have:

$$\lim_{n \to \infty} \left(\sqrt{\frac{1}{n} + x_n} + \sqrt{x_n}\right) = \lim_{n \to \infty} \left(\sqrt{\frac{1}{n}} + x_n\right) + \lim_{n \to \infty} \sqrt{x_n} = \sqrt{0 + 1 + \sqrt{1}} = 2.$$ 

So we get:

$$B = \lim_{n \to \infty} \frac{1}{\sqrt{n} \left(\sqrt{\frac{1}{n} + x_n} + \sqrt{x_n}\right)} = \frac{1}{2} \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0.$$ 

So $A = 1$ and $B = 0$ and we are done.