Theoretical Mathematics, Exam 1, 3/5/14

Name:

Show your work.
Twenty points per question.
The best five questions will count.

Question 1

For any positive integer $n$, put:

$$s_n = \frac{1}{3(5)} + \frac{1}{5(7)} + \frac{1}{7(9)} + \cdots + \frac{1}{(2n + 1)(2n + 3)},$$

$$t_n = \frac{n}{3(2n + 3)}.$$

Prove that $s_n = t_n$, for all $n \in \mathbb{N}$. 
Question 2

Let $x$ be the binary number $x = 0.0110\overline{0110}$.
(The digits under the bar of this number repeat forever).

- Write $x$ as a fraction.
- Find the infinite expansion of $x$ in base seven.
  Is this expansion unique?
  Discuss.
Question 3

Using the axioms for the field $\mathbb{Q}$ of rational numbers, solve the following equations.
Each step must be justified either by appeal to one of the axioms, a definition, or by using a theorem proved in class.
Basic properties of the integers may be assumed.

- $2x - 4 > 5 - x$

- $x^2 - 7x + 6 = 0$
Question 4

Solve, with proof, the inequality \(-1 < \frac{2x + 1}{3x - 1} < 1\) for the real unknown \(x\).
Write your solution as a union of open real intervals.
Question 5

Let $A = \{(x, y) \in \mathbb{R}^2 : y^2 - x^2 = 9\}$.

- Sketch the set $A$ in the Cartesian plane.

- Prove that the set $A$ is not a function.

- Construct, with proof, two different functions, each with domain the closed interval $[-4, 4]$, each of whose graphs is a subset of $A$ and exactly one of which has an inverse function.
  For each of your functions determine its range, with proof.
Question 6

Let $x_n = \frac{n^2}{(n + 1)^2}$, for any positive integer $n$.

Let $\mathcal{X} = \{x_n : n \in \mathbb{N}\}$.

- Prove that $\sup(\mathcal{X}) = 1$.

- Also find a real number $N$, such that $x_n$ is within $10^{-2}$ of 1, for all integers $n > N$. 

Question 7

For any $n \in \mathbb{N}$, let $J_n = \left[ -\frac{1}{n}, \frac{2n+1}{2n-1} \right]$.

- Sketch the intervals $J_1, J_2$ and $J_3$.
- Prove that $\{J_n : n \in \mathbb{N}\}$ is a nested sequence of intervals.
- Determine, with proof, the sets $A = \bigcap_{n=1}^{\infty} J_n$ and $B = \bigcup_{n=1}^{\infty} J_n$. 
Question 8

Let $A = \mathbb{R} - \{1\}$ and $B = \mathbb{R} - \{4\}$.

For $x \in A$, let $f(x) = \frac{4x - 3}{x - 1}$.

- Prove that $f$ gives a well-defined map from $A$ to $B$.
- Discuss, with proof, whether or not $f$ is invertible and if $f$ is invertible, obtain a formula for the inverse function.
- Determine, with proof, the direct image $f([2, \infty))$. 
Question 9

Let $A = \{2m - 1 : m \in \mathbb{Z}\}$.
Let $B = \{5n : n \in \mathbb{N}\}$.
Construct, with proof, a bijection between $A$ and $B$. 