L’Hôpital’s rule: a proof

Let \( f \) and \( g \) be defined and differentiable on \((a, b) \subset \mathbb{R}\) with \( a < b \). Here \( b \) could be \( \infty \).

- Suppose that \( g'(x) \neq 0 \), for any \( x \in (a, b) \).
- Suppose that \( f(x) \to 0 \) and \( g(x) \to 0 \) as \( x \to b^- \). Alternatively suppose that \( |g(x)| \to \infty \), as \( x \to b^- \).
- Suppose that \( \frac{f'(x)}{g'(x)} \to L \) as \( x \to b^- \).

Then we have:

\[
\lim_{x \to b^-} \frac{f(x)}{g(x)} = L.
\]

Proof:

By the Intermediate Value Property of the derivative, \( g'(x) \) always has the same sign on \((a, b)\), so \( g \) is strictly monotonic on \((a, b)\) by the Mean Value Theorem.

In particular we have \( g(x) \neq g(t) \) if \( x \neq t \) and \( x \) and \( t \) lie in \((a, b)\). Also since \( g(x) \) is strictly monotonic, \( g(x) \) can be zero at at most one point, \( c \), say, with \( a < c < b \) again by the Mean Value Theorem.

By shrinking the interval \((a, b)\) to the interval \((c, b)\), if necessary, we may assume henceforth that \( g(x) \) is everywhere non-zero on the interval \((a, b)\).

Let \( \epsilon > 0 \) be given.

Since \( \lim_{u \to b^-} \frac{f'(u)}{g'(u)} = L \), there is real number \( a_\epsilon \) in \((a, b)\) such that, for any \( z \in (a_\epsilon, b) \), we have:

\[
\left| \frac{f'(z)}{g'(z)} - L \right| < \frac{\epsilon}{2}.
\]

Fix \( x \) in \((a_\epsilon, b)\).

By the Cauchy Mean Value Theorem, we have, for any \( t \in (x, b) \), that a real number \( c(x, t) \) exists with \( a_\epsilon < x < c(x, t) < t < b \), such that:

\[
\frac{f(x) - f(t)}{g(x) - g(t)} = \frac{f'(c(x, t))}{g'(c(x, t))}.
\]
In particular for any \( t \in (x, b) \), we have \( c(x, t) \in (a_\varepsilon, b) \) also.
So we have for any given \( x \in (a_\varepsilon, b) \), the relation, valid for any \( x < t < b \):

\[
\left| \frac{f(x) - f(t)}{g(x) - g(t)} - L \right| < \frac{\varepsilon}{2}.
\]

- First we consider the case that \( f(u) \to 0 \) and \( g(u) \to 0 \) as \( u \to b^- \).
  Take the limit of this relation as \( t \to b^- \), giving for each \( x \in (a_\varepsilon, b) \):
  \[
  \left| \frac{f(x)}{g(x)} - L \right| = \left| \frac{f(x) - 0}{g(x) - 0} - L \right| = \lim_{t \to b^-} \left| \frac{f(x) - f(t)}{g(x) - g(t)} - L \right| \leq \frac{\varepsilon}{2} < \varepsilon.
  \]
  So \( \lim_{x \to b^-} \frac{f(x)}{g(x)} = L \) and we are done.

- Now we consider the case that \( |g(t)| \to \infty \) as \( t \to \infty \).

By the above work, we have, for any given (fixed) real \( x \in (a_\varepsilon, b) \), the relation, valid for any real \( t \) such that \( x < t < b \):

\[
\left| \frac{f(x) - f(t)}{g(x) - g(t)} - L \right| = \left| \frac{\frac{f(t)}{g(t)} - \frac{f(x)}{g(x)}}{1 - \frac{g(x)}{g(t)}} - L \right| < \frac{\varepsilon}{2},
\]

\[
-\frac{\varepsilon}{2} \left| 1 - \frac{g(x)}{g(t)} \right| < \frac{f(t)}{g(t)} - L - \left( \frac{f(x) - Lg(x)}{g(t)} \right) < \frac{\varepsilon}{2} \left| 1 - \frac{g(x)}{g(t)} \right|,
\]

\[
-\frac{\varepsilon}{2} \left| 1 - \frac{g(x)}{g(t)} \right| + \left( \frac{f(x) - Lg(x)}{g(t)} \right) < \frac{f(t)}{g(t)} - L < \frac{\varepsilon}{2} \left| 1 - \frac{g(x)}{g(t)} \right| + \left( \frac{f(x) - Lg(x)}{g(t)} \right),
\]

As \( t \to b^- \), for fixed \( x \in (a_\varepsilon, b) \), since \( |g(t)| \to \infty \), whereas the terms \( f(x), g(x) \) and \( L \) are fixed and bounded, the quantity \( \left| 1 - \frac{g(x)}{g(t)} \right| \to 1 \) and the quantity \( \left( \frac{f(x) - Lg(x)}{g(t)} \right) \to 0 \), so the left hand side of the above inequality is larger than \( -\varepsilon \) and the right hand side is less than \( \varepsilon \), for \( a_\varepsilon < x < \delta_\varepsilon < t < b \).

So we have, for \( \delta_\varepsilon < t < b \):

\[
\left| \frac{f(t)}{g(t)} - L \right| < \varepsilon.
\]

So \( \lim_{t \to \infty} \frac{f(t)}{g(t)} = L \), as required and we are done.