Real analysis, Math 1530 Midterm exam 10/25/13
Name:

Question 1:
True/false questions; ten points each; the best six count
For each of the following statements, determine with proof whether or not they are valid:

• The boundary of the standard Cantor set is countable.

• Let $f : \mathbb{Q} \to \mathbb{Q}$ be continuous.
  Then there is a unique continuous function $g : \mathbb{R} \to \mathbb{R}$ such that we have $g(x) = f(x)$ for all $x \in \mathbb{Q}$.

• For $X$ and $Y$ topological spaces, let $f : X \to Y$ be a bijection.
  Then $f$ is a homeomorphism if and only if $f$ is open (maps open sets in $X$ to open sets in $Y$).
True/false questions; ten points each; the best six count

• Let \( X = \{0, 1\} \).
  If \( \tau \) is a topology for \( X \), then \( X \) is connected, unless \( \tau \) is the discrete topology.

  

• If \( S \subseteq \mathbb{R} \), then \( S \) is connected if and only if \( S \) is path connected.

  

• If \( S \subseteq X \), with \( X \) a metric space, then if \( S \) is path connected, so is \( \overline{S} \).
True/false questions; ten points each; the best six count

- There exist a pair of homeomorphic metric spaces, one of which is bounded the other unbounded.

- Let $\mathcal{S} = \{s_n : n \in \mathbb{N}\} \subset \mathbb{R}$ be a sequence.
  Put $\mathcal{T} = \{|s_n| : n \in \mathbb{N}\}$.
  Then $\mathcal{S}$ has a convergent subsequence if and only if $\mathcal{T}$ has a convergent subsequence.
The best three additional questions count: twenty points each

Question 2

Let \( f : \mathbb{R} \to [-1, 1] \) be given as follows:

- \( f(0) = 0 \),
- \( f(x) = \sin(x^{-1}) \), when \( 0 \neq x \in \mathbb{R} \).

Prove that \( f \) is not continuous.

Is \( f \) an open map: i.e. if \( U \) is open in \( \mathbb{R} \) is \( f(U) \) open in \([-1, 1]\)?

Prove your answer.
Question 3

Let \( g : \mathbb{R} \to \mathbb{R} \) be given by as follows:

\begin{itemize}
  \item \( g(0) = 1 \)
  \item \( g(x) = \cos(x^{-2}) \) when \( 0 \neq x \in \mathbb{R} \).
\end{itemize}

Let \( f : \mathbb{R} \to \mathbb{R} \) be continuous.

Prove that the product function \( h = fg \) is continuous if and only if \( f(0) = 0 \).

If now \( f(0) = 0 \) and \( f \) is uniformly continuous, is \( h \) also uniformly continuous?

Explain your answer.
Question 4

Let $\mathcal{C}$ be the standard Cantor set.

For $\alpha \in \mathcal{C}$, write $\alpha = 0.a_1a_2\ldots a_n\ldots$, its base three expansion, where for each $j \in \mathbb{N}$, $a_j$ is either 0 or 2.

Then put $f(\alpha) = 0.b_1b_2\ldots b_n\ldots$, the binary number for which for each $j \in \mathbb{N}$, $b_j = 1$ if and only if $a_j = 2$ and $b_j = 0$ if and only if $a_j = 0$.

Prove that the map $f$ gives a continuous surjection from $\mathcal{C}$ to $[0,1]$.

If $f(\beta) = f(\gamma)$ for $\beta$ and $\gamma$ in $\mathcal{C}$, what can one say about the numbers $\beta$ and $\gamma$?

Explain your answer.
Question 5

Prove that $[0,1]$ cannot be written as a countably infinite union of disjoint closed subintervals.
Question 6

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and surjective.
Prove that $f$ is open (maps open subsets of the reals to open subsets) if and only if $f$ is an homeomorphism.
Question 7

Define a function $f : \mathbb{R}^2 \to \mathbb{R}$ as follows:

- $f(0, 0) = 0,$
- For $(0, 0) \neq (x, y) \in \mathbb{R}^2$, $f(x, y) = \frac{xy}{x^2 + y^2}$.

Prove the following:

- For any fixed real $s$ the function $g : \mathbb{R} \to \mathbb{R}$ given by the formula $g(y) = f(s, y)$, for each $y \in \mathbb{R}$, is continuous.
- For any fixed real $t$ the function $h : \mathbb{R} \to \mathbb{R}$ given by the formula $h(x) = f(x, t)$, for each $x \in \mathbb{R}$ is continuous.
- $f$ is not continuous.
Question 8

Let $C$ be a countably infinite subset of a metric space. Suppose that the diameter of $C$ is finite. Prove or disprove that $C$ is compact if and only if $C$ is closed.