Theoretical Mathematics, Quiz 4 Solutions, 11/4/11

Question 1

Let \( x_n = \frac{2n^2 + 3}{2n^2 - 1} \), defined for any \( n \in \mathbb{N} \).

Show that the sequence \( x_n \) is monotone.

Also find, with proof, the supremum and infimum of the sequence.

Hence, or otherwise, find, with proof, \( \lim_{n \to \infty} x_n \).

There are many ways to prove monotonicity; we give some examples.

- Prove that \( x_n - x_m \) has the same or the opposite sign as does \( n - m \):

  We have, for any \( m \) and \( n \) in \( \mathbb{N} \):

  \[
  x_n - x_m = \frac{2n^2 + 3}{2n^2 - 1} - \frac{2m^2 + 3}{2m^2 - 1} = \frac{(2n^2 + 3)(2m^2 - 1) - (2m^2 + 3)(2n^2 - 1)}{(2n^2 - 1)(2m^2 - 1)}
  \]

  \[
  = \frac{4n^2m^2 + 6m^2 - 2n^2 - 3 - (4m^2n^2 + 6n^2 - 2m^2 - 3)}{(2n^2 - 1)(2m^2 - 1)}
  \]

  \[
  = \frac{8m^2 - 8n^2}{(2n^2 - 1)(2m^2 - 1)} = (n - m) \left( -\frac{8(m + n)}{(2n^2 - 1)(2m^2 - 1)} \right).
  \]

  Now the quantity \( \left( -\frac{8(m + n)}{(2n^2 - 1)(2m^2 - 1)} \right) \) is always negative, for any \( m \) and \( n \) in \( \mathbb{N} \), so we see that \( x_n - x_m \) and \( n - m \) have opposite signs, so the sequence \( x_n \) is strictly decreasing, so is monotone, as required.

- Rewrite \( x_n \).

  For any \( n \in \mathbb{N} \), we have:

  \[
  x_n = \frac{2n^2 + 3}{2n^2 - 1} = \frac{2n^2 - 1 + 4}{2n^2 - 1} = 1 + 4 \left( \frac{1}{2n^2 - 1} \right).
  \]

  Now \( n^2 \) increases with \( n \), so \( 2n^2 - 1 \) is positive and strictly increases with \( n \), so \( \frac{1}{2n^2 - 1} \) strictly decreases with \( n \), so the sequence \( x_n \) is strictly decreasing.
• Prove that \( x_n - x_{n+1} \) has a fixed sign:

For any \( n \in \mathbb{N} \), we have:

\[
x_n - x_{n+1} = \frac{2n^2 + 3}{2n^2 - 1} - \frac{2(n + 1)^2 + 3}{2(n + 1)^2 - 1}
\]

\[
= \frac{(2n^2 + 3)(2(n + 1)^2 - 1) - (2(n + 1)^2 + 3)(2n^2 - 1)}{(2n^2 - 1)(2(n + 1)^2 - 1)}
\]

\[
= \frac{(4n^2(n + 1)^2 + 6(n + 1)^2 - 2n^2 - 3 - (4n^2(n + 1)^2 + 6n^2 - 2(n + 1)^2 - 3))}{(2n^2 - 1)(2(n + 1)^2 - 1)}
\]

\[
= \frac{8((n + 1)^2 - n^2)}{(2n^2 - 1)(2(n^2 + 2n + 1) - 1)}
\]

\[
= \frac{8(2n + 1)}{(2n^2 - 1)(2n^2 + 4n + 1)} > 0.
\]

So the sequence \( x_n \) is strictly decreasing.

• Write \( x_n \) as a subsequence of a simpler monotonic sequence.

For any \( n \in \mathbb{N} \), we have:

\[
x_n = \frac{2n^2 + 3}{2n^2 - 1} = \frac{2n^2 - 1 + 4}{2n^2 - 1} = y_{2n^2 - 1},
\]

\[
y_n = \frac{n + 4}{n} = 1 + \frac{4}{n}.
\]

Clearly \( y_n \) is strictly decreasing, since \( n^{-1} \) strictly decreases with \( n \).

So \( x_n \), being a well-defined subsequence of the monotonic decreasing sequence \( y_n \) (since the subscript \( 2n^2 - 1 \) strictly increases with \( n \)), is itself strictly decreasing.
Since the sequence \( x_n \) is strictly decreasing, we have:

\[
\sup(\{x_n : n \in \mathbb{N}\}) = x_1 = 5.
\]

We note that \( x_n > 1 \), for all \( n \in \mathbb{N} \), since \( x_n \) is a quotient of two positive numbers, with the denominator smaller than the numerator.

So the sequence \( x_n \) is bounded below by 1, so its infimum \( \beta \) exists and is at least 1.

We claim that \( \beta = \inf(\{x_n : n \in \mathbb{N}\}) = 1. \)

Suppose that \( \beta > 1 \).

Then for all \( n \in \mathbb{N} \), we have:

\[
\frac{2n^2 + 3}{2n^2 - 1} \geq \beta,
\]

\[
\frac{2n^2 + 3}{2n^2 - 1} = \frac{2n^2 - 1 + 4}{2n^2 - 1} = 1 + \frac{4}{2n^2 - 1} \geq \beta,
\]

\[
\frac{4}{2n^2 - 1} \geq \beta - 1 > 0,
\]

\[
\frac{2n^2 - 1}{4} \leq \frac{1}{\beta - 1}
\]

\[
2n^2 \leq 1 + \frac{4}{\beta - 1} = \frac{\beta + 3}{\beta - 1},
\]

\[
n^2 \leq \frac{\beta + 3}{2(\beta - 1)},
\]

\[
n \leq \sqrt{\frac{\beta + 3}{2(\beta - 1)}}.
\]

So \( \mathbb{N} \) is bounded above, contrary to the Archimedean Principle.

So \( \beta > 1 \) leads to a contradiction.

So \( \beta = 1 \) and the sequence \( \{x_n ; n \in \mathbb{N}\} \) has infimum 1.

For the last part, we have since \( x_n \) is a bounded monotone decreasing sequence:

\[
\lim_{n \to \infty} x_n = \inf(\{x_n : n \in \mathbb{N}\}) = 1.
\]
Question 2

Find, with proof, the following limits:

- \[ \lim_{n \to \infty} n \left( \sqrt{n^2 + 4} - \sqrt{n^2 + 2} \right) \]
  
  We have:
  \[
  \lim_{n \to \infty} n \left( \sqrt{n^2 + 4} - \sqrt{n^2 + 2} \right) = \lim_{n \to \infty} n \left( \frac{\left( \sqrt{n^2 + 4} \right)^2 - \left( \sqrt{n^2 + 2} \right)^2}{\sqrt{n^2 + 4} + \sqrt{n^2 + 2}} \right) \]
  \[
  = \lim_{n \to \infty} n \left( \frac{n^2 + 4 - n^2 - 2}{\sqrt{n^2 + 4} + \sqrt{n^2 + 2}} \right) = \lim_{n \to \infty} \frac{n^2 + 4 - (n^2 + 2)}{\sqrt{n^2 + 4} + \sqrt{n^2 + 2}} \]
  \[
  = \lim_{n \to \infty} \frac{2n}{\sqrt{n^2 + 4} + \sqrt{n^2 + 2}} = \lim_{n \to \infty} \frac{2}{\sqrt{n^2 + 4} + \sqrt{n^2 + 2}} \]
  \[
  = \lim_{n \to \infty} \left( \sqrt{(1 + 0) + \sqrt{(1 + 0)}} \right) = 1.
  \]

- \[ \lim_{n \to \infty} \frac{n(n + 1)}{2^n} \]

  Put \( x_n = \frac{n(n + 1)}{2^n} \), for any \( n \in \mathbb{N} \).
  
  Note that \( x_n > 0 \), for all \( n \in \mathbb{N} \).
  
  Now we have, for any \( n \in \mathbb{N} \):
  \[
  \frac{x_{n+1}}{x_n} = \left( \frac{(n + 1)(n + 2)}{2^{n+1}} \right) \left( \frac{2^{\frac{3}{2}}}{n(n + 1)} \right) \]
  \[
  = \left( \frac{n + 2}{n \left( \frac{1}{2} \right)} \right) = \frac{1}{\sqrt{2}} \left( 1 + \frac{2}{n} \right).
  \]
  
  Taking the limit as \( n \to \infty \), we get:
  \[
  \lim_{n \to \infty} \left( \frac{x_{n+1}}{x_n} \right) = \frac{1}{\sqrt{2}} \lim_{n \to \infty} \left( 1 + \frac{2}{n} \right) = \frac{1}{\sqrt{2}} (1 + 0) = \frac{1}{\sqrt{2}} < 1.
  \]
  
  By the ratio test we conclude that \( \lim_{n \to \infty} x_n = 0 \).
  
  So the required limit exists and is zero.
Question 3

Let \( x_n = -\frac{1}{n} \) if \( n \) is odd and \( x_n = \frac{n+1}{n} \), if \( n \) is even.

Find, with proof, \( \lim(\sup(x_n)) \) and \( \lim(\inf(x_n)) \).

The even subsequence is monotone decreasing, since \( x_n = 1 + \frac{1}{n} \), when \( n \)

is even.

The odd subsequence is monotone increasing, since \( \frac{1}{n} \) decreases as \( n \)

increases.

Every term of the even subsequence is larger than every term of the odd

subsequence.

- If \( n \) is even, \( A_n = \sup(\{x_k : k \geq n\}) = \frac{n+1}{n} \).

- If \( n \) is odd, \( A_n = \sup(\{x_k : k \geq n\}) = x_{n+1} = \frac{n+2}{n+1} \).

- If \( n \) is odd, \( B_n = \inf(\{x_k : k \geq n\}) = x_n = -\frac{1}{n} \).

- If \( n \) is even, \( B_n = \inf(\{x_k : k \geq n\}) = x_{n+1} = -\frac{1}{n+1} \).

We have, given \( \epsilon > 0 \):

- For \( n \) even, \( |A_n - 1| = \left| \frac{n+1}{n} - 1 \right| = \frac{1}{n} < \epsilon \), when \( n > N_\epsilon = \frac{1}{\epsilon} \).

- For \( n \) odd, \( |A_n - 1| = \left| \frac{n+2}{n+1} - 1 \right| = \frac{1}{n+2} < \frac{1}{n} < \epsilon \), when \( n > N_\epsilon = \frac{1}{\epsilon} \).

So we have \( |A_n - 1| < \epsilon \), when \( n > N(\epsilon) \), so \( \lim(\sup(x_n)) = \lim_{n \to \infty} A_n = 1 \).

Also we have, given \( \epsilon > 0 \):

- For \( n \) odd, \( |B_n - 0| = |B_n| = \left| -\frac{1}{n} \right| = \frac{1}{n} < \epsilon \), when \( n > M_\epsilon = \frac{1}{\epsilon} \).

- For \( n \) even, \( |B_n - 0| = |B_n| = \left| -\frac{1}{n+1} \right| = \frac{1}{n+1} < \frac{1}{n} < \epsilon \), when \( n > M_\epsilon = \frac{1}{\epsilon} \).

So we have \( |B_n - 0| < \epsilon \), when \( n > M(\epsilon) \), so \( \lim(\inf(x_n)) = \lim_{n \to \infty} B_n = 0 \).
Question 4

Let a recursion be given by the formula:

\[ x_{n+1} = \frac{3 + x_n}{1 + x_n}, \quad x_1 = 1. \]

Find \( x_2, x_3 \) and \( x_4 \) and prove that the sequence \( x_n \) has a limit \( L \).

Also find, with proof, the value of \( L \).

Note that its is clear (by induction, IMP), that \( x_n > 0 \) for all \( n \in \mathbb{N} \).

So we may write \( x_{n+1} = f(x_n) \), where \( f(x) = \frac{3 + x}{1 + x} \), which is well-defined and positive for any \( x > 0 \), so \( f : \mathbb{R}^+ \to \mathbb{R}^+ \), where \( \mathbb{R}^+ \) is the set of all positive reals.

We have:

\[ x_2 = f(x_1) = f(1) = \frac{3 + 1}{1 + 1} = \frac{4}{2} = 2, \]

\[ x_3 = f(x_2) = f(2) = \frac{3 + 2}{1 + 2} = \frac{5}{3}, \]

\[ x_4 = f(x_3) = f(2) = \frac{3 + \frac{5}{3}}{1 + \frac{5}{3}} = \frac{9 + 5}{5 + 3} = \frac{14}{8} = \frac{7}{4}. \]

For \( x \) and \( y \) positive, with \( x \neq y \), we have:

\[ f(x) - f(y) = \frac{3 + x}{1 + x} - \frac{3 + y}{1 + y} = \frac{(3 + x)(1 + y) - (3 + y)(1 + x)}{(1 + x)(1 + y)} \]

\[ = \frac{3 + 3y + x + xy - (3 + 3x + y + xy)}{(1 + x)(1 + y)} \]

\[ = (x - y) \left( \frac{-2}{(1 + x)(1 + y)} \right). \]

So \( f(x) - f(y) \) has the opposite sign as does \( x - y \), so \( f \) is strictly decreasing.

So \( g = f \circ f \) is strictly increasing and the even and odd subsequences of \( x_n \) are strictly monotonic.

Since \( x_3 = \frac{5}{3} > x_1 = 1 \), the odd subsequence is strictly increasing.

Also, since \( x_4 = \frac{7}{4} < x_2 = 2 \), the even subsequence is strictly decreasing.
Now if \( x_n < x_{n+1} \), we have \( g(x_n) < g(x_{n+1}) \), but \( g(x_n) = x_{n+2} \) and \( g(x_{n+1}) = x_{n+3} \), so \( x_{n+2} < x_{n+3} \).

Since \( 1 = x_1 < x_2 = 2 \), we have by induction (IMP) that \( x_n < x_{n+1} \), for all odd positive integers \( n \).

But if \( n \) is odd, \( n + 1 \) is even, so \( x_{n+1} \leq x_2 = 2 \), so \( x_n < x_{n+1} \leq 2 \).

So \( x_n < 2 \) for all odd integers \( n \).

Also if \( n \) is a positive even integer, then \( n-1 \) is positive and odd, so \( x_{n-1} < x_n \).

But \( x_{n-1} \geq x_1 = 1 \). So \( x_n > x_{n-1} \geq 1 \).

So \( x_n > 1 \), for all even integers \( n \).

Let \( y_n = \{x_{2n-1} : n \in \mathbb{N}\} \) and \( z_n = \{x_{2n} : n \in \mathbb{N}\} \). Then we have shown that \( y_n \) is a strictly increasing sequence, bounded above by 2 and below by \( y_1 = x_1 = 1 \).

So \( y = \lim_{n \to \infty} y_n \) exists and \( 1 \leq y \leq 2 \). Also we have shown that \( z_n \) is a strictly decreasing sequence, bounded below by 1 and above by \( z_1 = x_2 = 2 \).

So \( z = \lim_{n \to \infty} z_n \) exists and \( 1 \leq z \leq 2 \).

Now we have the recursions, valid for all \( n \in \mathbb{N} \):

\[
y_{n+1} = x_{2n+1} = (f \circ f)(x_{2n-1}) = g(x_{2n-1}) = g(y_n), \quad z_{n+1} = x_{2n+2} = (f \circ f)(x_{2n}) = g(z_n).
\]

Next, we have, for any real \( x > 0 \):

\[
g(x) = (f \circ f)(x) = f(f(x)) = \frac{3 + f(x)}{1 + f(x)} = \frac{3 + \left( \frac{3 + x}{1 + x} \right)}{1 + \left( \frac{3 + x}{1 + x} \right)} = \frac{3(1 + x) + 3 + x}{1 + x + 3 + x} = \frac{6 + 4x}{4 + 2x} = \frac{3 + 2x}{2 + x}.
\]

So now we have:

\[
y = \lim_{n \to \infty} y_n = \lim_{n \to \infty} y_{n+1} = \lim_{n \to \infty} g(y_n) = \lim_{n \to \infty} \left( \frac{3 + 2y_n}{2 + y_n} \right) = \frac{3 + 2y}{2 + y},
\]

\[
y(2 + y) = 3 + 2y, \quad 2y + y^2 = 3 + 2y, \quad y^2 = 3, \quad y = \pm \sqrt{3}.
\]

Since \( y \geq 1 \), we have \( y = \sqrt{3} \).

Next we have:

\[
z = \lim_{n \to \infty} z_n = \lim_{n \to \infty} z_{n+1} = \lim_{n \to \infty} g(z_n) = \lim_{n \to \infty} \left( \frac{3 + 2z_n}{2 + z_n} \right) = \frac{3 + 2z}{2 + z},
\]

\[
z(2 + z) = 3 + 2z, \quad 2z + z^2 = 3 + 2z, \quad z^2 = 3, \quad z = \pm \sqrt{3}.
\]

Since \( z \geq 1 \), we have \( z = \sqrt{3} \).

Finally, since \( y = z = \sqrt{3} \), the given sequence also has the same limit, so \( \lim_{n \to \infty} x_n = \sqrt{3} \) and \( L = \sqrt{3} \).