The story of $\sqrt{2}$: Pythagoras and Dedekind
Pythagoras: the square root of 2 does not exist!

We prove the following theorem:

- Let \( x \) be a rational number.
  
  Then \( x^2 \neq 2 \).

Proof:

A rational number is one of the form \( \frac{p}{q} \), where \( p \) and \( q \) are integers and \( q \neq 0 \). So the theorem to be proved is equivalent to the following theorem:

- The only solution to the equation \( x^2 = 2y^2 \), with \( x \) and \( y \) integers is 
  
  \( x = y = 0 \).

Since \( (-x)^2 = x^2 \), for \( x \in \mathbb{Z} \), we may rephrase the theorem to be proved as:

- Let \( x \) and \( y \) be positive integers.
  
  Then \( x^2 \neq 2y^2 \).

Proof:

Let \( S = \{ s : s = p + q \text{ and } (p, q) \in \mathbb{N} \times \mathbb{N} \text{ and } p^2 = 2q^2 \} \).

We must show that \( S \) is the empty set.

If \( S \) is not the empty set, then since \( S \subseteq \mathbb{N} \), by the Well Ordering Principle, POW, the set \( S \) has a least element, \( k \), say and there exist positive integers \( t \) and \( u \) such that \( t^2 = 2u^2 \) and \( t + u = k \).

Note that \( k \geq 3 \), since if \( k \leq 2 \), we would have to have \( k = 2 \) and then we would have to have \( t = u = 1 \); but then \( t^2 = 2u^2 \) is false.

Note that \( t^2 \) is even.

Then \( t \geq 2 \), since if \( t = 1 \), then \( t^2 = 1 \), which is not even.

But \( t^2 - t = t(t-1) \) is also even, being the product of two consecutive positive integers. So \( t = t^2 - (t^2 - t) \) is also even.

So we may write \( t = 2r \), where \( r \in \mathbb{N} \).

Then we have \( 2u^2 = t^2 = (2r)^2 = 2(2r^2) \), so \( u^2 = 2r^2 \).

Then \( p = u \) and \( q = r \) solves the equation \( p^2 = 2q^2 \), with \( p \) and \( q \) positive integers.

This gives the element \( s = p + q = u + r \) of \( S \).

But then \( s = u + r < u + r + r = u + 2r = u + t = k \).

So \( s < k \), which contradicts the definition of \( k \).

So the hypothesis that \( S \) is non-empty leads to a contradiction.

Therefore \( S \) is the empty set and we are done.
Note that we used the following properties of evenness:

- A positive integer $n$ is even if and only if $n = 2m$ for some positive integer $m$.

This is just the definition of evenness of positive integers.

- 1 is not even.

Proof:
If $1 = 2m$, with $m \in \mathbb{N}$, then we have $m \geq 1$, so $1 = 2m \geq 2$, so $1 \geq 2$, which is false.

- If $a$ and $b$ are even positive integers, then $a + b$ is even.

Proof:
We have $a = 2c$ and $b = 2d$, for $c$ and $d$ in $\mathbb{N}$.
Then $a + c = 2c + 2d = 2(c + d)$ and $c + d \in \mathbb{N}$, so $a + c$ is even.

- If $a$ and $b$ are even positive integers, and $a > b$, then $a - b$ is even.

Proof:
We have $a = 2c$ and $b = 2d$, for integers $c$ and $d$.
If now $d \geq c$, then $2d \geq 2c$, so $b \geq a$, which is false, so $c > d$. 
So $c - d \in \mathbb{N}$ and then we have $a - b = 2c - 2d = 2(c - d)$, so $a - b$ is even.

- If $n$ is a positive integer, then $s(n) = n(n + 1)$ is even.

Proof:
We use induction.

- When $n = 1$, we have $s(1) = 1(1 + 1) = 1(2) = 2(1)$, so $s(1)$ is even, the base case.

- If $s(n)$ is even, so $n(n + 1) = 2t$, for $t \in \mathbb{N}$, then we have:

$$s(n + 1) = (n + 1)(n + 1 + 1) = (n + 1)(n + 2)$$

$$= (n + 1)n + (n + 1)(2) = s(n) + 2(n + 1) = 2t + 2(n + 1) = 2(t + n + 1).$$

So $s(n + 1)$ is even, since $t + n + 1 \in \mathbb{N}$.

So by IMP, the Principle of Mathematical Induction, we are done.
Dedekind: the square root of 2 does exist!

We prove the following theorem:

- There is a unique positive real number, \( x \), such that \( x^2 = 2 \).
  Also \( 1 < x < 2 \).

Proof:

We observe first that if \( 0 < x < y \), with \( x \) and \( y \) real, then \( 0 < x^2 < xy \) and \( 0 < xy < y^2 \), so \( 0 < x^2 < y^2 \).

So if \( x \geq 2 \), then \( x^2 \geq 4 > 2 \), so \( x^2 > 2 \) and \( x^2 = 2 \) is false.

Also if \( 0 < x \leq 1 \), then \( 0 < x^2 \leq 1 < 2 \), so \( x^2 < 2 \) and \( x^2 = 2 \) is false.

So if \( x \) is real and \( x^2 = 2 \), with \( x > 0 \), then \( 1 < x < 2 \), as required.

Also if \( x^2 = 2 \) and \( y^2 = 2 \), with \( x > 0 \) and \( y > 0 \), then we have:

\[
0 = x^2 - y^2 = (x - y)(x + y) \text{ and } x + y > 0, \text{ so } x = y.
\]

So if \( x > 0 \) and \( x^2 = 2 \), then \( 1 < x < 2 \) and \( x \) is unique as required.

It remains to prove that there is a real \( x \) such that \( 1 \leq x \leq 2 \) and \( x^2 = 2 \).

To do this, we define two sets:

\[ S = \{ s \in \mathbb{R} : 1 \leq s \leq 2 \text{ and } s^2 < 2 \}, \]

\[ T = \{ t \in \mathbb{R} : 1 \leq t \leq 2 \text{ and } t^2 > 2 \}. \]

Then if \( s \in S \) and \( t \in T \), we have \( t^2 > 2 > s^2 \), so \( t^2 - s^2 > 0 \), so we have:

\((t - s)(t + s) > 0.\)

But \( t + s \geq 1 + 1 = 2 > 0 \), so \( t - s > 0 \) also and \( t > s \).

Also \( 2 \in T \) and \( 1 \in S \), so \( S \) is non-empty and bounded above by \( 2 \), so \( \alpha = \sup(S) \) exists.

Also, since \( S \subset [1, 2] \), we have \( 1 \leq \alpha \leq 2 \).

Also \( T \) is non-empty and bounded below by \( 1 \), so \( \beta = \inf(T) \) exists.

Also, since \( T \subset [1, 2] \), we have \( 1 \leq \beta \leq 2 \).

Since \( s < t \), for any \( s \in S \) and \( t \in T \), any fixed \( t \in T \) is an upper bound for \( S \), so we have \( \alpha \leq t \), for any \( t \in T \).

This means that \( \alpha \) is a lower bound for \( T \), so we get \( \alpha \leq \beta \).

So we have shown that \( 1 \leq \alpha \leq \beta \leq 2 \).
We next show that $\alpha \beta = 2$.

- If $s \in \mathbb{S}$, then $1 \leq s \leq 2$ and $s^2 < 2$.
  
  Put $t = \frac{2}{s}$.
  
  Since $0 < 1 \leq s \leq 2$, we have $\frac{1}{2} \leq \frac{1}{s} \leq 1$, so $1 \leq \frac{2}{s} \leq 2$, so $1 \leq t \leq 2$.
  
  Also $t^2 = \frac{4}{s^2} > \frac{4}{2} = 2$.
  
  So $t^2 > 2$.
  
  So $t \in \mathbb{T}$, so $t \geq \beta$.

  So $\frac{2}{s} \geq \beta$, so $s \leq \frac{2}{\beta}$.

  So $\frac{2}{\beta}$ is an upper bound for $\mathbb{S}$.

  So $\alpha \leq \frac{2}{\beta}$, so $\alpha \beta \leq 2$.

- If $t \in \mathbb{T}$, then $1 \leq t \leq 2$ and $t^2 > 2$.
  
  Put $s = \frac{2}{t}$.
  
  Since $0 < 1 \leq t \leq 2$, we have $\frac{1}{2} \leq \frac{1}{t} \leq 1$, so $1 \leq \frac{2}{t} \leq 2$, so $1 \leq s \leq 2$.

  Also $s^2 = \frac{4}{t^2} < \frac{4}{2} = 2$.

  So $s^2 < 2$.

  So $s \in \mathbb{S}$, so $s \leq \alpha$.

  So $\frac{2}{t} \leq \alpha$, so $t \geq \frac{2}{\alpha}$.

  So $\frac{2}{\alpha}$ is a lower bound for $\mathbb{T}$.

  So $\beta \geq \frac{2}{\alpha}$, so $\alpha \beta \geq 2$.

Since $\alpha \beta \leq 2$ and $\alpha \beta \geq 2$, we have $\alpha \beta = 2$. 
Summarizing, we have proved that both $\alpha = \sup(\mathbb{S})$ and $\beta = \inf(\mathbb{T})$ exist and obey:

$$1 \leq \alpha \leq \beta \leq 2, \quad \alpha \beta = 2.$$  

- If now $\alpha = \beta$, we are done, since then $\alpha \beta = \alpha^2 = 2$.

- If instead, $\alpha \neq \beta$, then $\alpha < \beta$.
  
  Let a real number $u$ be chosen, so that $\alpha < u < \beta$.
  
  Then $u$ exists: for example, we may take $u = \frac{\alpha + \beta}{2}$.
  
  Then since $\alpha \geq 1$ and $\beta \leq 2$, we have $1 \leq \alpha < u < \beta \leq 2$, so $1 \leq u \leq 2$.

  - If $u^2 < 2$, then $u \in \mathbb{S}$, by definition of $\mathbb{S}$.
    
    So $u \leq \alpha$, by definition of $\alpha$, which contradicts that $u > \alpha$.
  
  - If $u^2 > 2$, then $u \in \mathbb{T}$, by definition of $\mathbb{T}$.
    
    So $u \geq \beta$, by definition of $\beta$, which contradicts that $u < \beta$.

So $u^2 < 2$ and $u^2 > 2$ each leads to a contradiction, so by trichotomy $u^2 = 2$ and we are done.

Actually, the last case, $\alpha < \beta$, cannot occur, since if $\alpha < \beta$, we may choose two numbers $u$ and $v$, such that $1 \leq \alpha < u < v < \beta \leq 2$:

For example take $u = \frac{1}{3}(2\alpha + \beta)$ and $v = \frac{1}{3}(\alpha + 2\beta)$.

Then the argument given above, applied separately to each of $u$ and $v$, shows that $u^2 = v^2 = 2$, so $(u - v)(u + v) = u^2 - v^2 = 0$, so $u - v = 0$, since $u + v \geq 2 > 0$, so $u = v$, which contradicts that $u < v$. So $\alpha < \beta$ is impossible.

Alternatively in the argument given above, if $\alpha < \beta$, by the density theorem, we may choose $u$ to be rational and $\alpha < u < \beta$.

Then the argument above shows that $u^2 = 2$, and $u$ is rational, contradicting Pythagoras. So $\alpha < \beta$ is impossible.

Since $\alpha < \beta$ is false, we have $\alpha = \beta$ and we have proved the additional result, pinning down the square root of two:

- We have $\sup(\mathbb{S}) = \inf(\mathbb{T}) = \sqrt{2}$.

This formula allows us to systematically calculate $\sqrt{2}$ to any required degree of precision.

Finally note that we did not need to use the Archimedean Principle in the above proof.