Theoretical Mathematics, Quiz 1 Solutions, 1/11/12

Question 1

Let $A = \{x \in \mathbb{R}; -8 \leq x^3 \leq 8\}$.
Let $B = \{x \in \mathbb{R}; 0 < x^2 < 9\}$.

Sketch the sets $A \cup B$, $A \cap B$, $A - B$ and $B - A$ on the real line.

Also sketch the set of all pairs $(x, y)$ with $x \in A$ and $y \in B$ in the Cartesian plane.

- For $x$ real, we have $-8 \leq x^3 \leq 8$ if and only if $-2 \leq x \leq 2$.
  So $A = [-2, 2]$, a closed interval.

- For $x$ real, we have $0 < x^2 < 9$ if and only if $0 < |x| < 3$, off and only if $0 < x < 3$ or $-3 < x < 0$.
  So $B = (-3, 0) \cup (0, 3)$ a union of open intervals.
  So $B$ is open.

Then we have:

- $A \cup B = [-2, 2] \cup ((-3, 0) \cup (0, 3)) = \{0\} \cup B = (-3, 3)$, since the only element of $A$ that is not in $B$ is 0. The set $A \cup B$ is open.

- $A \cap B = [-2, 2] \cap ((-3, 0) \cup (0, 3)) = (-2, 2] \cap (-3, 0) \cup (-2, 2] \cap (0, 3) = [-2, 0) \cup (0, 2]$. The set $A \cap B$ is neither open nor closed.

- $A - B = \{0\}$, since the only element of $A$ that is not in $B$ is 0. The set $A - B$ is closed.

- $B - A = ((-3, 0) - [-2, 2]) \cup ((0, 3) - [-2, 2]) = (-3, -2) \cup (2, 3)$. The set $B - A$, being a union of open intervals, is open.

Finally the set of all pairs $(x, y)$ with $x \in A$ and $y \in B$ is the set of all pairs $(x, y)$ with $-2 \leq x \leq 2$ and $-3 < y < 0$ together with the set of all pairs $(x, y)$ with $-2 \leq x \leq 2$ and $0 < y < 3$, so is the union of two (solid) rectangles $ABCD$ and $BCEF$, where $A = (-2, -3)$, $B = (-2, 0)$, $C = (2, 0)$, $D = (2, -3)$, $E = (2, 3)$ and $F = (-2, 3)$. Here the left and right edges of each rectangle are included (minus their endpoints), but the bottom and top edges are excluded. None of the vertices of the rectangles is included.
Question 2

For \( n \) a positive integer, let \( s_n = 1^2 + 2^2 + 3^2 + \cdots + n^2 \) (\( n \) terms in the sum).

Prove that for any \( n \in \mathbb{N} \), we have the formula: \( s_n = \frac{1}{6} n(n + 1)(2n + 1) \).

We use induction (IMP).

- We have \( s_1 = 1^2 = 1 \) and when \( n = 1 \), we have \( \frac{1}{6} n(n + 1)(2n + 1) = \frac{1}{6}(1)(2)(3) = 1 \), so the base case holds.

- For the inductive step, we assume that for some \( k \in \mathbb{N} \) we have the formula \( s_k = \frac{1}{6} k(k + 1)(2k + 1) \).

Then we need to prove that:

\[
\begin{align*}
    s_{k+1} &= \frac{1}{6} (k + 1)((k + 1) + 1)(2(k + 1) + 1) \\
&= \frac{1}{6} (k + 1)(k + 2)(2k + 3).
\end{align*}
\]

But we notice that the sum \( s_{k+1} \) has all the terms of the sum \( s_k \) in its sum and one extra term, namely \((k + 1)^2\). Therefore, using the inductive hypothesis, we have:

\[
    s_{k+1} = s_k + (k + 1)^2
\]

\[
    = \frac{1}{6} k(k + 1)(2k + 1) + (k + 1)(k + 1)
\]

\[
    = \frac{1}{6} (k + 1) (k(2k + 1) + 6(k + 1))
\]

\[
    = \frac{1}{6} (k + 1)(2k^2 + k + 6k + 6)
\]

\[
    = \frac{1}{6} (k + 1)(2k^2 + 7k + 6)
\]

\[
    = \frac{1}{6} (k + 1)(k + 2)(2k + 3).
\]

This completes the inductive step.

So, by IMP, the required result is true for all \( n \in \mathbb{N} \): \( s_n = \frac{1}{6} n(n + 1)(2n + 1) \).
Question 3

Let $S$ and $T$ be sets. Prove that $S = T$ if and only if $S \cap T = S \cup T$.

We know that by the definition of union and intersection, we have:

\[
S \cap T \subset S, \quad S \cap T \subset T, \\
S \subset S \cup T, \quad T \subset S \cup T.
\]

- So if we have: $S \cap T = S \cup T$, we get first:

\[
S \subset S \cup T = S \cap T \subset T, \\
S \subset T.
\]

Also, we have:

\[
T \subset S \cup T = S \cap T \subset S, \\
T \subset S.
\]

Since $S \subset T$ and $T \subset S$ both hold, we get $S = T$, as required.

- Conversely, if $S = T$, we have $S \cap T = S \cap S = S$ and $S \cup T = S \cup S = S$, so $S \cap T = S \cup T$, since both sides of this equation are equal to $S$.

So we have proved the required result: $S = T$ if and only if $S \cap T = S \cup T$ and we are done.
Question 4

Prove the de Morgan distributive law $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, valid for any sets $A$, $B$ and $C$.

- Let $x \in A \cap (B \cup C)$.
  Then we have:
  - $x \in A$ and $x \in B \cup C$,
  - $x \in A$ and ($x \in B$ or $x \in C$),
  - $(x \in A$ and $x \in B$) or $(x \in A$ and $x \in C$),
  - $(x \in A \cap B)$ or $(x \in A \cap C)$,
  - $x \in (A \cap B) \cup (A \cap C)$.
  This proves that $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$.

- Let $y \in (A \cap B) \cup (A \cap C)$.
  Then we have:
  - $y \in A \cap B$ or $y \in A \cap C$,
  - $(y \in A$ and $y \in B$) or $(y \in A$ and $y \in C$),
  - $y \in A$ and ($y \in B$ or $y \in C$),
  - $y \in A \cap (B \cup C)$.
  This proves that $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$.

Since we have proved both that $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$ and $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$, we have $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and we are done.