Introduction to Theoretical Mathematics, Winter 2012
Homework Assignments
Homework 1, due Thursday 12th January

Lebl problems
Do exercises 0.3.1, 0.3.5, 0.3.6, 0.3.7, 0.3.8, 0.3.9, 0.3.11 and 0.3.13 of Lebl.

W problem 1
The W problem for this week is: 0.3.5 of Lebl.
Note this problem will not be scored as a homework problem.
Homework 2, due Thursday 19th January

Leibniz problems

Do exercises 0.3.3, 0.3.4, 0.3.10, 0.3.12, 0.3.15, 0.3.16 and 0.3.17 of Leibniz.

W problem 2

Write out the details of the proof that every positive integer is even or odd but not both, using the definition that an integer $n$ is odd if $n = 2p - 1$, for $p$ an integer, whereas an integer $m$ is even if $m = 2q$, for $q$ an integer.

You will need to use induction, make sure that you use it clearly.

Use your results to show that every integer is even or odd but not both.
Homework 3, due Thursday 26th January

Leibniz problems

Do exercises 0.3.14, 0.3.18, 0.3.19 and 0.3.20 of Leibniz.
Also do the following problem 0.3.21.

Problem 0.3.21

Prove that for each positive integer \( n \), the polynomial \( x^n - y^n \) is exactly divisible by \( x - y \).

Problem 0.3.22

Let \( s_n = 1(2) + 2(3) + 3(4) + \cdots + n(n + 1) \), where there are \( n \) terms in the sum, defined for each positive integer \( n \). Compute \( s_n \) for small \( n \).
Hence conjecture a formula for the sum and prove it by induction.

W Problem 3

Let \( A = \mathbb{R} - \{1\} \).
Let \( B = \mathbb{R} - \{2\} \).

Let \( f(x) = \frac{2x - 1}{x - 1} \), defined for any real \( x \in A \).

Prove that \( f : A \rightarrow B \) is well-defined and gives a bijection from \( A \) to \( B \) and find a formula for the inverse function \( f^{-1} \) of \( f \).
Also determine, with proof, the sets \( f((3, \infty)) \) and \( f^{-1}((3, \infty)) \).
Homework 4, due Thursday 2nd February

Leibniz Problems

In Leibniz, do exercises 0.4.3, 0.4.4, 0.4.5 and 0.4.6, 1.1.2, 1.1.3 and 1.1.4.
Also do the following problem 0.4.7.

Problem 0.4.7

In the Cartesian plane, let $S = \{(x, y) \in \mathbb{R}^2 : y^2 = x + 9\}$.

- Sketch the set $S$ and prove that $S$ is not a function.
- Find three different subsets of $S$, each of which is a function, each with domain the closed interval $[0, 16]$.
  For each of your three functions determine, with proof, its range.
- Suppose that a subset of $S$ is a function.
  Prove that the function is injective.

W Problem 4

Let $f : A \to B$ and $g : B \to A$ be maps.

- Suppose that $f \circ g = \text{id}_B$.
  Prove that $g$ is injective and $f$ is surjective.
- Suppose that $f \circ g = \text{id}_B$ and $g \circ f = \text{id}_A$.
  Prove that $f$ and $g$ are bijective and that $g$ is the inverse of $f$.
- Suppose that $g$ is injective.
  Prove that $f$ can be chosen so that $f \circ g = \text{id}_B$.
- Suppose that $f$ is surjective.
  Prove that $g$ can be chosen so that $f \circ g = \text{id}_B$. 
Homework 5, due Thursday 9th February

Leibniz Problems
In Leibniz, do exercises 1.1.1, 1.1.5, 1.1.6, 1.1.7.
Also do the following problems 1.1.8 and 1.1.9:

Problem 1.1.8
Let \( \mathbb{Z}_2 = \{0, 1\} \).

- Define addition in the set \( \mathbb{Z}_2 \) by the formulas:
  \[
  0 + 0 = 0, \quad 0 + 1 = 1 + 0 = 1, \quad 1 + 1 = 0.
  \]

- Define multiplication in the set \( \mathbb{Z}_2 \) by the formulas:
  \[
  0(0) = 0, \quad 0(1) = 1(0) = 0, \quad 1(1) = 1.
  \]

Prove that \( \mathbb{Z}_2 \), with the given rules for multiplication and division, is a field. The quadratic equations with coefficients in \( \mathbb{Z}_2 \) are \( x^2, x^2 + x, x^2 + 1 \) and \( x^2 + x + 1 \).
Which of these can be factored as products of the linear polynomials \( x \) and \( x + 1 \)?
Explain your answer.

Problem 1.1.9
Let \( \mathbb{G} = (0, 1] \subset \mathbb{R} \).
Define multiplication for \( \mathbb{G} \) by the same formula as one uses for real numbers.
Define the sum \( a + b \) of elements of \( \mathbb{G} \) by the same formula as one uses for real numbers, unless the ordinary real sum exceeds 1, in which case one subtracts 1.
So, for example, we have \( \frac{1}{2} + \frac{1}{3} = \frac{5}{6} \), but \( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2} - 1 \).

Which axioms of a field are violated by the set \( \mathbb{G} \)?
Explain your answer.
W Problem 5

Let \( F = (a, b) : a \in \mathbb{Z}_2 \) and \( b \in \mathbb{Z}_2 \).
Define addition component-wise:

\[
(a, b) + (c, d) = (a + c, b + d), \text{ for any } a, b, c \text{ and } d \text{ in } \mathbb{Z}_2.
\]

So for example we have:

\[
(0, 1) + (1, 1) = (0 + 1, 1 + 1) = (1, 0).
\]

Define multiplication by the formula:

\[
(a, b).(c, d) = (ac + bd, bc + ad + bd), \text{ for any } a, b, c \text{ and } d \text{ in } \mathbb{Z}_2.
\]

So for example we have:

\[
(0, 1).(1, 1) = (0(1) + 1(1), 1(1) + 0(1) + 1(1)) = (0 + 1, 1 + 0 + 1) = (1, 0).
\]

- Construct the addition and multiplication tables for \( F \) and prove that \( F \) with the given operations, is a field.
- Show that \( F \) contains a copy of \( \mathbb{Z}_2 \) as a subfield.
- Find all solutions \( x \in F \) of the polynomial equation \( x^2 + x + 1 = 0 \) and hence factorize this polynomial as a product of linear factors (whose coefficients are in \( F \)).
Homework 6 , due Thursday 16th February

Lebl Problems

In Lebl, do exercises 1.2.1, 1.2.2, 1.2.3, 1.2.4, 1.2.5, 1.2.6 and 1.2.7.
For question 1.2.2, also prove that the number $n \in \mathbb{N}$ that you find is unique.

W Problem 6

In the Cartesian plane, let $\mathcal{S} = \{(x, y) \in \mathbb{R}^2 : y^2 = x + 16\}$.

- Sketch the set $\mathcal{S}$ and prove that $\mathcal{S}$ is not a function.
- Find three different subsets of $\mathcal{S}$, each of which is a function, with domain $[0, 9]$.
- For each of your functions determine, with proof, its range.
- If $T \subset \mathcal{S}$ is a function with domain a set $A \subset \mathbb{R}$ and range a set $B \subset \mathbb{R}$ does the function $T : A \rightarrow B$ have an inverse?
  Discuss.
Homework 7, due Thursday 23rd February

Lcl Problems
In Lebl, do exercises 1.2.8, 1.2.9, 1.2.10, 1.2.11, 1.3.1, 1.3.2, 1.3.3 and 1.3.5.

W Problem 7
Let $S = \left\{ \frac{2n - 1}{n + 2} : n \in \mathbb{N} \right\}$ and let $T = \left\{ \frac{2n^2 - 1}{n^2 + 2} : n \in \mathbb{N} \right\}$.
Prove that $\sup(S) = \sup(T)$ and $\inf(S) = \inf(T)$.
Homework 8, due Thursday 15th March

Lebl Problems
In Lebl, do exercises 1.4.1, 1.4.2, 1.4.3, 1.4.4, 1.4.5, 1.5.1, 1.5.2 and 1.5.3.

W Problem 8
Let $\mathcal{I} = I_n = \left[ \frac{n^2 - 1}{n^2 + 1}, \frac{n^2 + 1}{n^2 - n + 1} \right] : n \in \mathbb{N}.$

Prove that $\mathcal{I}$ is a nested sequence of closed intervals and determine, with proof, the union and common intersection of the members of the sequence.
Homework 9, due Thursday 22nd March

Lebl Problems
In Lebl, do exercises 1.6.1, 1.6.2, 2.1.1-2.1.8.

W Problem 9

The Fibonacci sequence is defined by $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$, for any integer $n$ with $n \geq 2$.
Let $\alpha$ and $\beta$ be the two roots of the polynomial equation $x^2 - x - 1 = 0$, with $\alpha > \beta$.
Prove that $f_n \sqrt{5} = \alpha^n - \beta^n$, for any positive integer $n$.
Hence, or otherwise, determine, with proof, the limit $\lim_{n \to \infty} \frac{f_{n+1}}{f_n}$.
For $n$ a non-negative integer, put $s_n(t) = \sum_{k=0}^{n} f_k t^k$, where $t$ is a real number. Compute $s_n(t)$ in terms of $\alpha$ and $\beta$ and determine, with proof, the limit $s(t) = \lim_{n \to \infty} s_n(t)$ including a determination of the values of $t$, for which the limit $s(t)$ exists.
Homework 10, due Thursday 29th March

Lebl Problems

Prepare for the exam Wednesday: everything about the reals, sup and inf of ordered sets and sequences to the end of section 2.1.
In Lebl, do exercises 2.1.9-2.16.

W Problem 10

A sequence $X = \{x_n : n \in \mathbb{N}\}$ is defined recursively by the formulas:

$$x_1 = 2, \quad x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}.$$

Prove that $x_n > \sqrt{2}$, for all $n \in \mathbb{N}$.
Also prove that the sequence $X$ is monotonic and determine, with proof, its limit, $x$, say.
Also estimate the difference $x_{10} - x$. 

Homework 11, due Thursday 5th April

Lebl Problems
Do problems 2.2.3-2.2.9 of Lebl.

W Problem 11
Write an essay on the use of induction in mathematics, including examples of the use of each kind of induction.
Homework 12, due Thursday 12th April

Lebl Problems
Do problems 2.3.5, 2.3.7, 2.3.9, 2.4.4, 2.4.5, 2.5.3, 2.5.4, 2.5.7 of Lebl.

W Problem 12
Write an essay on the use of Reductio ad Absurdum in mathematics, including examples where it seems the use is essential, and others where it is convenient but can be circumvented.
Homework 13, due Thursday 19th April

Lebl Problems
Do problems 2.6.1, 2.6.2, 2.6.3, 2.6.4, 2.7.1 and 2.7.2, of Lebl.

W Problem 13
Write an essay on some facet of mathematical history, discussing how it is relevant to your own mathematical experience.