Theoretical Mathematics, Final Examination, 4/27/12

Name:

Show your work.
Twenty points per question.
The best six questions will count, with up to ten bonus points for a seventh question.

Question 1

For each positive integer \( n \), let \( s_n \) be given by the following sum:

\[
s_n = 1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2,
\]

where there are \( n \) terms in the sum.

Prove that, for each positive integer \( n \), we have:

\[
s_n = \frac{1}{3}n(2n - 1)(2n + 1).
\]

Also determine, with proof, the following limits, or explain why the limit in question does not exist:

\[
\text{• } A = \lim_{n \to \infty} \left( \frac{s_n}{n(n + 1)(n + 2)} \right)
\]

\[
\text{• } B = \lim_{n \to \infty} \left( \frac{s_{2n+1} - s_{2n}}{4n^2 - 1} \right)
\]
Question 2

Let $\mathbb{A} = \mathbb{R} - \{2\}$ and let $\mathbb{B} = \mathbb{R} - \{3\}$.

For $x \in \mathbb{A}$, let $f(x) = \frac{3x - 2}{x - 2}$.

- Prove that $f$ is a well-defined map from $\mathbb{A}$ to $\mathbb{B}$.
- Discuss, with proof, whether or not $f$ is invertible and if $f$ is invertible, obtain a formula for the inverse function.
- Determine, with proof, the direct image $f([4, \infty))$. 
Question 3

Let $f : A \to B$ be an injection.

- Suppose that $A$ is countably infinite.  
  Prove that there is an injection from $B$ to $A$ if and only if $B$ is countably infinite.

- Suppose instead that $A$ is uncountable.  
  Can $B$ be countable? Explain your answer.
Question 4

Using the ordering axioms for the reals, or relevant theorems on ordering properties of the reals, solve the following inequalities. For each, write your solution as a union of intervals. Also, for each solution, discuss, with proof, whether or not your solution can be written as a union of open intervals.

- $|x^2 - 7x + 2| < 10$

- $\sqrt{\frac{3x - 2}{x - 2}} > 1$
Question 5

Let $x$ be the base seven number $x = 0.250_7$.

Write $x$ as a fraction: $x = \frac{p}{q}$, where $p$ and $q$ are positive integers, with no common factors.
Also write $x$ as a decimal and in base two.
Are the decimal and base two representations of $x$ unique? Discuss.
Question 6

Let \( x_n = \frac{(n + 1)(n + 2)}{(n - 1)(n - 2)} \), defined for any integer \( n \geq 3 \).

- Prove that the sequence \( x_n \) is monotonic.
- Prove, from first principles, that \( \lim_{n \to \infty} x_n = 1 \).
- Also find, with proof, a real number \( N \) such that \( x_n \) is within \( 10^{-10} \) of 1, for all integers \( n > N \).
Question 7

Let $\mathbb{A}$ and $\mathbb{B}$ be sets of reals, such that between any two distinct elements of $\mathbb{A}$ there is an element of $\mathbb{B}$ and vice-versa, between any two distinct elements of $\mathbb{B}$ there is an element of $\mathbb{A}$.

- Prove that if $\mathbb{A}$ is finite, then so is $\mathbb{B}$ and the the number of elements of the set $\mathbb{A}$ differs from the number of elements of the set $\mathbb{B}$ by at most one.

- Give an example (with proof) where $\mathbb{A}$ is countable and $\mathbb{B}$ is uncountable and $\mathbb{A} \cap \mathbb{B} = \emptyset$.

- Suppose that $\mathbb{A}$ is infinite and bounded.
  Prove that $\mathbb{B}$ is also infinite and bounded.
  Suppose also that $\sup(\mathbb{A}) \notin \mathbb{A}$ and $\sup(\mathbb{B}) \notin \mathbb{B}$.
  Prove that $\sup(\mathbb{A}) = \sup(\mathbb{B})$. 
Question 8

Determine, with proofs, the following limits, or prove that the limit in question does not exist:

\[ \lim_{n \to \infty} \left( \frac{(n - 1)^2(n^2 + 1)}{(n^3 - 2)\sqrt{n^2 + n + 1}} \right) \]

\[ \lim_{n \to \infty} n \left( \sqrt{n^2 + 1} - \sqrt{n^2 - 1} \right) \]

\[ \lim_{n \to \infty} \left( 1 + \frac{2}{n} \right)^{2n} \]

Hint: first write \( 1 + \frac{2}{n} = \left(1 + \frac{1}{n+1}\right)\left(1 + \frac{1}{n}\right) \).
Question 9

Let \( x_n = -\left(\frac{n + 2}{n + 1}\right) \), if \( n \) is odd and \( x_n = \frac{1}{\sqrt{n}} \) if \( n \) is even.

Let \( X = \{x_n : n \in \mathbb{N}\} \).
For any \( k \in \mathbb{N} \), put \( X_k = \{x_n : n \geq k, n \in \mathbb{N}\} \).

Let \( a_k = \inf(X_k) \) and \( b_k = \sup(X_k) \), for any \( k \in \mathbb{N} \).
Put \( J_k = [a_k, b_k] \), for any \( k \in \mathbb{N} \).

Prove that \( \{J_k : k \in \mathbb{N}\} \) is a nested sequence of intervals.
Do there exist three subsequences of \( X \) with different limits?
Explain your answer.
Question 10

Let $X = \{x_n : n \in \mathbb{N}\}$ be a sequence of positive real numbers, such that $X$ has no least element.

Prove that $X$ has a monotonic convergent subsequence.

Also give an example, with proof, of such a sequence $X$, that has at least two monotonic convergent subsequences, whose limits are different.
Question 11

Determine with proof, whether or not each of the following series converges and if it converges, decide, with proof, whether or not the convergence is absolute.

Also, if a series converges, either determine its sum exactly, or give an estimate of the sum, with proof, accurate to within an error of at most \( \frac{1}{4} \):

- \( A = \sum_{n=0}^{\infty} \left( \frac{(-5)^n}{3^{2n}} \right) \)

- \( B = \sum_{n=1}^{\infty} \left( (-1)^n \sqrt{\frac{n}{4n^2 + 1}} \right) \)

- \( C = \sum_{n=1}^{\infty} \left( \frac{2n^2}{n^n} \right) \)