Theoretical Mathematics, Quiz 4 Solutions, 11/9/10

Question 1

Let \( A = \left\{ \frac{2n^2 + 1}{2n^2 - 1}; \ n \in \mathbb{N} \right\} \).

Compute the elements of \( A \) for the cases \( n = 1, 2, 3 \).

Find, with proof, the supremum and infimum of \( A \).

For \( n \in \mathbb{N} \), put \( z_n = \frac{2n^2 + 1}{2n^2 - 1} \).

We have:

\[
z_1 = \frac{2(1) + 1}{2(1) - 1} = 3, \quad z_2 = \frac{2(4) + 1}{2(4) - 1} = \frac{9}{7}, \quad z_3 = \frac{2(9) + 1}{2(9) - 1} = \frac{19}{17}.
\]

We suspect that \( z_n \) is decreasing.

- We have:
  \[
z_n = \frac{2n^2 + 1}{2n^2 - 1} = \frac{(2n^2 - 1) + 2}{2n^2 - 1} = 1 + \frac{2}{2n^2 - 1}.
  \]

Now \( n^2 \) increases with \( n \), so \( 2n^2 - 1 \) increases also, so \( (2n^2 - 1)^{-1} \) decreases, as does \( 2(2n^2 - 1)^{-1} \). So \( z_n \) decreases with \( n \).

- Alternatively we compute:

\[
\begin{align*}
  z_{n+1} - z_n &= \frac{2(n + 1)^2 + 1}{2(n + 1)^2 - 1} - \frac{2n^2 + 1}{2n^2 - 1} \\
  &= \frac{(2(n + 1)^2 + 1)(2n^2 - 1) - (2(n + 1)^2 - 1)(2n^2 + 1)}{(2(n + 1)^2 - 1)(2n^2 - 1)} \\
  &= \frac{4n^2(n + 1)^2 + 2n^2 - 2(n + 1)^2 - 4n^2(n + 1)^2 - 2n^2 + 2(n + 1)^2 - 1}{(2(n + 1)^2 - 1)(2n^2 - 1)} \\
  &= \frac{4n^2 - 4(n + 1)^2}{(2(n + 1)^2 - 1)(2n^2 - 1)} \\
  &= \frac{4n^2 - 4n^2 - 4n - 4}{(2(n + 1)^2 - 1)(2n^2 - 1)} \\
  &= -\frac{(8n + 4)}{(2(n + 1)^2 - 1)(2n^2 - 1)} < 0.
\end{align*}
\]

So the sequence \( z_n \) decreases with \( n \).
Since \( z_n \) is a decreasing sequence, we have:

\[
\sup(A) = z_1 = 3.
\]

Next we have:

\[
z_n - 1 = \frac{2n^2 + 1 - 1}{2n^2 - 1} = \frac{2n^2 + 1 - (2n^2 - 1)}{2n^2 - 1} = \frac{2}{2n^2 - 1} > 0.
\]

So \( z_n \geq 1 \) and 1 is a lower bound for the sequence \( z_n \).
So \( z = \inf(A) \) exists and we have \( z \geq 1 \).
Suppose that \( z > 1 \).
Then, since \( z \) is a lower bound for the sequence \( z_n \), we have for all \( n \in \mathbb{N} \):

\[
1 < z \leq z_n,
\]

\[
0 < z - 1 \leq z_n - 1 = \frac{2}{2n^2 - 1},
\]

\[
0 < \frac{2n^2 - 1}{2} \leq \frac{1}{z - 1},
\]

\[
0 < 2n^2 - 1 \leq \frac{2}{z - 1},
\]

\[
0 < 2n^2 \leq \frac{2}{z - 1} + 1 = \frac{2 + z - 1}{z - 1} = \frac{z + 1}{z - 1},
\]

\[
0 < n^2 \leq \frac{z + 1}{2(z - 1)},
\]

\[
n \leq \sqrt{\frac{z + 1}{2(z - 1)}}.
\]

So \( N \) is bounded above, contradicting Archimedes.
So \( z > 1 \) is false.
Since \( z \geq 1 \), we have \( z = 1 \) and we have proved that \( \inf(A) = 1 \).
Question 2

- Find the base ten expansion of $x = \frac{3}{11}$.

We have:

$$3 = 0(11) + 3,$$
$$30 = 2(11) + 8,$$
$$80 = 7(11) + 3,$$
$$30 = 2(11) + 8, \ldots$$

At each stage we divide by 11 and then multiply the remainder by the base 10, and iterate.
The quotients give the successive digits of the expansion.
We see that the result is repetitive, and we have:

$$\frac{3}{11} = 0.\overline{27}.$$

Check: if $u = 0.\overline{27}$, then we have:

$$100u = 27.\overline{27} = 27 + u,$$
$$99u = 27,$$
$$u = \frac{27}{99} = \frac{(3)(9)}{(11)(9)} = \frac{3}{11}.$$
Find the base three expansion of $y = \frac{4}{7}$.

We have:

$$
4 = 0(7) + 4, \\
12 = 1(7) + 5, \\
15 = 2(7) + 1, \\
3 = 0(7) + 3, \\
9 = 1(7) + 2, \\
6 = 0(7) + 6, \\
18 = 2(7) + 4, \\
12 = 1(7) + 5, \ldots
$$

At each stage we divide by 7 and then multiply the remainder by the base 3, and iterate. The quotients give the successive digits of the expansion. We see that the result is repetitive, and we have:

$$
\frac{4}{7} = 0.\overline{120102}_3.
$$

Check: if $v = 0.\overline{120102}$ in base 3, then we have:

$$
3^6v = 120102.\overline{120102} = 120102_3 + v,
$$

$$
729v = 3^5 + 2(3^4) + 3^2 + 2 + v,
$$

$$
728v = 243 + 162 + 9 + 2 = 416,
$$

$$
v = \frac{416}{728} = \frac{(4)(104)}{(7)(104)} = \frac{4}{7}.
$$

If the binary expansion of a real number $z$ begins $z = 0.0111 \ldots$, which smallest closed real interval can we be sure that $z$ lies in? Explain. Write your answer in the form $[a, b]$ where $a$ and $b$ are fractions.

We have $0.0111\overline{0} \leq z \leq 0.0111\overline{1} = 0.1\overline{0}.

Now we have $0.0111\overline{0} = 2^{-2} + 2^{-3} + 2^{-4} = \frac{1}{16}(4 + 2 + 1) = \frac{7}{16}$ and $0.1\overline{0} = 2^{-1} = \frac{1}{2}$.

Therefore we can be sure that $z$ lies in the closed interval $\left[\frac{7}{16}, \frac{1}{2}\right]$. 
Question 3

For \( n \in \mathbb{N} \), let \( J_n \) be the closed interval: 
\[
J_n = \left[ -\frac{1}{2n}, \frac{n+3}{n+2} \right].
\]
Sketch the intervals \( J_1, J_2 \) and \( J_3 \).

Prove that the intervals \( \{J_n; n \in \mathbb{N}\} \) form a nested sequence of intervals.
Determine, with proof, the sets \( J = \cap_{n=1}^{\infty} J_n \) and \( K = \cup_{n=1}^{\infty} J_n \), in each case writing your answer as an appropriate interval.

We have:
\[
J_1 = \left[ -\frac{1}{2}, \frac{4}{3} \right], \quad J_2 = \left[ -\frac{1}{4}, \frac{5}{4} \right], \quad J_3 = \left[ -\frac{1}{6}, \frac{6}{5} \right].
\]

For \( n \in \mathbb{N} \), put \( x_n = -\frac{1}{2n} \) and \( y_n = \frac{n+3}{n+2} \).
Then for each \( n \in \mathbb{N} \), we have \( J_n = [x_n, y_n] \). Note that for each \( n \in \mathbb{N} \), we have \( x_n < 0 < 1 < y_n \), so \( x_n < y_n \) and \([0, 1] \subset J_n \), so \([0, 1] \subset J\).

- As \( n \) increases, so does \( 2n \).
  Then \( \frac{1}{2n} \) decreases, so \( -\frac{1}{2n} \) increases.
Alternatively, we have, for any \( n \in \mathbb{N} \):
\[
x_{n+1} - x_n = \frac{-1}{2n + 2} + \frac{1}{2n} = \frac{2n + 2}{2n(2n + 2)} = \frac{1}{2n(n+1)} > 0.
\]
So the sequence \( x_n \) is increasing.

- We have:
\[
y_n = \frac{n + 3}{n + 2} = \frac{(n + 2) + 1}{n + 2} = 1 + \frac{1}{n+2}.
\]
Now as \( n \) increases, so does \( n+2 \), so \( (n+2)^{-1} \) decreases, so \( y_n \) decreases with \( n \).
Alternatively, we have, for any \( n \in \mathbb{N} \)
\[
y_{n+1} - y_n = \frac{n + 4}{n + 3} - \frac{n + 3}{n + 2} = \frac{(n + 4) + 1}{(n + 2)(n + 3)} = \frac{1}{(n+2)(n+3)} < 0.
\]
So the sequence \( y_n \) is decreasing.
Since $x_n \leq y_n$ and $x_n$ is increasing, whereas $y_n$ is decreasing, the intervals $J_n$ form a nested sequence of closed intervals.

In particular $J_n \subset J_1$ for all $n \in \mathbb{N}$. So $K = \bigcup_{n=1}^{\infty} J_n \subset J_1$. But by definition of $K$, we have $J_1 \subset K$. Since $J_1 \subset K \subset J_1$, we have:

$$K = J_1 = \left[ -\frac{1}{2}, \frac{4}{3} \right].$$

Finally, by the theory of nested intervals, proved in class, we have:

$$J = [x, y], \quad x = \sup(x_n), \quad y = \inf(y_n).$$

Above we noted that $[0, 1] \subset J$, so $x \leq 0$ and $y \geq 1$.

- If $x < 0$, then for all $n \in \mathbb{N}$ we have, since $x$ is an upper bound for the sequence $x_n$:

$$x_n \leq x < 0,$$

$$-\frac{1}{2n} \leq x < 0,$$

$$0 < -x < \frac{1}{2n},$$

$$0 < 2n < -\frac{1}{x},$$

$$0 < n < -\frac{1}{2x}.$$  

So $\mathbb{N}$ is bounded above, contradicting Archimedes. So $x < 0$ is false. Since $x \leq 0$, we get $x = 0$.

- If $y > 1$, then for all $n \in \mathbb{N}$ we have, since $y$ is a lower bound for the sequence $y_n$:

$$0 < y - 1 \leq y_n - 1 = \frac{n+3}{n+2} - 1 = \frac{n + 3 - (n + 2)}{n + 2} = \frac{1}{n + 2},$$

$$0 < n + 2 < \frac{1}{y - 1},$$

$$n < \frac{1}{y - 1} - 2 = \frac{1 - (2(y - 1))}{y - 1} = \frac{3 - 2y}{y - 1}.$$  

So $\mathbb{N}$ is bounded above, contradicting Archimedes. So $y > 1$ is false. Since $y \geq 1$, we get $y = 1$.

So we have proved that $J = [x, y] = [0, 1]$ and we are done.