Complex variables: Quiz 2 Solutions 7/2/9

Question 1

Sketch the following sets in the complex plane and for each identify whether the set is open, closed or neither and whether or not the set is connected or compact. For each of these sets also give a parametrization or parametrizations of its boundary, as appropriate.

- \( \mathbb{A} = \{ z : 2 \Re(z) \geq \Im(z) \} \).
  Writing \( z = x + iy \), with \( x \) and \( y \) real we need \( 2x \geq y \).
  So the region is the closed half-plane below and to the right of the line \( y = 2x \).
  The region is closed, unbounded, non-compact and connected.
  The boundary is the line \( y = 2x \), so is parametrized by \( (x, y) = -t(1, 2) \)
  and so \( z = x + iy = -t - 2it = -t(1 + 2i) \).
  Here the parameter \( t \) ranges over all real numbers.
  The minus signs in the parametrization come from the fact that we want to traverse the boundary from top to bottom, since we want to trace it counter-clockwise with respect to an observer in the region \( \mathbb{A} \).

- \( \mathbb{B} = \{ z : |z| < 2 \text{ and } |z - 2| < 2 \} \).
  This is the intersection of \( B(0, 2) \) and \( B(2, 2) \).
  These open balls are bounded by circles which meet where:
  \[
  |z|^2 = 4, \quad x^2 + y^2 = 4,
  \]
  \[
  |z - 2|^2 = 4, \quad (x - 2)^2 + y^2 = 4,
  \]
  Subtracting, we get \( 0 = x^2 - (x - 2)^2 = (x - (x - 2))(x + x - 2) = 4(x - 1) \),
  so \( x = 1 \) and then \( y^2 = 4 - x^2 = 4 - 1 = 3 \).
  So the circles meet at \( \left(1, \pm \frac{\sqrt{3}}{2}\right) \), so at \( z = 1 \pm i \sqrt{3} = 2e^{\pm \frac{\pi}{3}} \).
  The region is bounded on the right by the arc of the circle
  \[
  z = 2e^{is}, \quad s = \frac{\pi}{3} \ldots \frac{5\pi}{3},
  \]
  The region is bounded on the left by the arc of the circle:
  \[
  z = 2 + 2e^{it}, \quad t = \frac{2\pi}{3} \ldots \frac{4\pi}{3}.
  \]
  The region is open, bounded, non-compact and connected.
\[ C = \left\{ z = re^{i\theta} : 2 < r < 4 \text{ and } 0 \leq \theta \leq \frac{\pi}{2} \right\}. \]

The region lies in the first quadrant and is bounded by the circles center the origin and radius two (on the inside) and four (on the outside).

The region is neither open nor closed, is bounded, non-compact and connected.

The boundary is parametrized as:

\[
\begin{align*}
z &= 4e^{is}, \quad 0 \leq s \leq \frac{\pi}{2}, \\
z &= i(4 - t), \quad 0 \leq t \leq 2, \\
z &= 2e^{-iu}, \quad -\frac{\pi}{2} \leq u \leq 0, \\
z &= 2 + v, \quad 0 \leq v \leq 2.
\end{align*}
\]
Question 2
Consider the transformation $T : z \rightarrow iz + 4 - 2i$, defined for any $z \in \mathbb{C}$.

- Find and sketch the image under $T$ of the points $0, 1, i$ and $1 + i$. Also describe the transformation $T$ geometrically.

Put $A = (0, 0) = 0, B = (1, 0) = 1, C = (1, 1) = 1 + i, D = (0, 1) = i$. Then we get:

- $A' = T(0) = i(0) + 4 - 2i = 4 - 2i = (4, -2)$,
- $B' = T(1) = i(1) + 4 - 2i = i + 4 - 2i = 4 - i = (4, -1)$,
- $C' = T(1 + i) = i(1 + i) + 4 - 2i = i - 1 + 4 - 2i = 3 - i = (3, -1)$,
- $D' = T(i) = i(i) + 4 - 2i = -1 + 4 - 2i = 3 - 2i = (3, -2)$.

Plotting, we see that the points $ABCD$ are in order around a unit square and $A'B'C'D'$ are also in order around a unit square, but rotated through $\frac{\pi}{2}$ radians.

When we find the fixed point $(3, 1)$, see below, it follows that the transformation $T$ is a rotation through $\frac{\pi}{2}$ radians centered at the point $(3, 1)$ or $3 + i$.

- Find the fixed points of $T$ (so solve the equation $T(z) = z$).

We need:

$$z = T(z) = iz + 4 - 2i,$$
$$z(1 - i) = 4 - 2i,$$
$$z(1 - i)(1 + i) = (4 - 2i)(1 + i)$$
$$2z = 6 + 2i,$$
$$z = 3 + i.$$

Check:

$$T(3 + i) = i(3 + i) + 4 - 2i = 3i - 1 + 4 - 2i = 3 + i.$$ 

So the only fixed point of $T$ is $z = 3 + i$, or the Cartesian point $(3, 1)$.

Then the transformation $T$ is a rotation through $\frac{\pi}{2}$ radians centered at the point $(3, 1)$ or $3 + i$. 

3
• Find a formula for the inverse transformation (so solve the equation $w = T(z) = iz + 4 - 2i$ for $z$ in terms of $w$). If $w = iz + 4 - 2i$, then we get:

$$w - 4 + 2i = iz,$$

$$z = -i(w - 4 + 2i) = -iw + 2 - 4i.$$ 

So the inverse transformation is $T^{-1}(z) = -iz + 2 - 4i$, defined for any complex $z$.

Geometrically, the inverse transformation is the rotation clockwise about the point $(3, 1)$ through an angle of $\frac{\pi}{2}$ radians.
Question 3

Consider the function of the complex variable \( z \), \( f(z) = (1 + z)^{\frac{1}{2}} \).

Given that \( f(0) = 1 \) and that \( f(z) \) has a branch cut on the interval \((-\infty, -1]\) of the negative real axis, determine the values of:

- \( f(3) \)
- \( f(i) \)
- \( \lim_{t \to 0^+} f(-5 + ti) \)
- \( \lim_{t \to 0^-} f(-5 + ti) \)

The vector connecting the point \(-1\) to the point \( z \) is the vector \( 1 + z \).

We use the polar representation of this complex number:

\[
1 + z = se^{i\alpha}.
\]

Here \( s = |1 + z| \) is real and positive and \( \alpha \) radians is the angle made by the vector \( 1 + z \) with the positive \( x \)-axis.

Then \( (1 + z)^{\frac{1}{2}} = \sqrt{s} e^{i\frac{\alpha}{2}} \).

At \( z = 0 \), we have \( s = |1 + 0| = 1 \), so since we are given that \( f(0) = 1 \), we need \( e^{i\frac{\alpha}{2}} = 1 \), so we may take \( \alpha = 0 \).

Then, by continuity, the range of \( \alpha \) is \(-\pi < \alpha < \pi\) on the cut plane.

- At \( z = 3 \), we have \( s = |1 + 3| = 4 \) and \( \alpha = 0 \), so \( f(3) = \sqrt{4} = 2 \).
- At \( z = i \), we have \( s = |1 + i| = \sqrt{2} \) and \( \alpha = \frac{\pi}{4} \), so we get:

\[
f(i) = 2^{\frac{1}{2}} e^{i\frac{\pi}{4}} = 2^{\frac{1}{2}} \left( \cos \left( \frac{\pi}{8} \right) + i \sin \left( \frac{\pi}{8} \right) \right) = 1.098684113 + i 0.4550898606.
\]

We may compute \( e^{i\frac{\pi}{4}} \) algebraically:

\[
e^{i\frac{\pi}{4}} = x + iy, \quad x \text { and } y \text { real and positive,}
\]

\[
x^2 + y^2 = \left| e^{i\pi/4} \right|^2 = 1,
\]

\[
\frac{1}{\sqrt{2}}(1+i) = \cos \left( \frac{\pi}{4} \right) + i \sin \left( \frac{\pi}{4} \right) = e^{i\pi/4} = \left( e^{i\pi/4} \right)^2 = (x+iy)^2 = x^2 - y^2 + 2ixy.
\]

5
Taking real and imaginary parts, we have the following relations:

\[
x^2 + y^2 = 1, \quad x^2 - y^2 = \frac{1}{\sqrt{2}}, \quad 2xy = \frac{1}{\sqrt{2}}.
\]

Adding and subtracting, we get:

\[
2x^2 = 1 + \frac{1}{\sqrt{2}}, \quad 2y^2 = 1 - \frac{1}{\sqrt{2}},
\]

\[
4x^2 = 2 + \sqrt{2}, \quad 4y^2 = 2 - \sqrt{2},
\]

\[
x = \frac{1}{2} \sqrt{2 + \sqrt{2}}, \quad y = \frac{1}{2} \sqrt{2 - \sqrt{2}},
\]

\[
f(i) = 2^{\frac{i}{2}}(x + iy) = \frac{1}{2} \left( \sqrt{2 + \sqrt{2}} + i \sqrt{2 - \sqrt{2}} \right)
= 1.098684113 + i 0.4550898606.
\]

- As \( z \to -5 \), we have \( 1 + z \to -4 \), so \( s = |1 + z| \to 4 \) and \( s^{\frac{i}{2}} \to 2 \).

Approaching \( z = -5 \) from the upper half-plane, we get \( \alpha \to \pi \).

Approaching \( z = -5 \) from the upper half-plane, we get \( \alpha \to -\pi \).

So we have the required limits:

\[
\lim_{t \to 0^+} f(-5 + ti) = 2e^{\frac{i\pi}{2}} = 2i,
\]

\[
\lim_{t \to 0^+} f(-5 + ti) = 2e^{\frac{-i\pi}{2}} = -2i.
\]