Question 1

For each of the following Euclidean transformations, describe the transformation and describe its invariant points and lines, if any:

- \( \mathcal{R} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \)

We recognize the transformation \( \mathcal{R} \) as a translation, mapping the point \((x, y)\) to \((x - 2, y - 3)\).

As such there are no invariant points and the invariant lines are the lines parallel to the direction of the translation vector \((-2, -3)\). These are the lines of slope \(-\frac{3}{2}\), so have equation \(y = \frac{3}{2}x + c\), or \(3x - 2y + a = 0\), where \(a\) is an arbitrary real number.

So the lines with line co-ordinates \([3, -2, a]\) are the invariant lines.

We check the invariance:

\[
[3, -2, a] \mathcal{R}^{-1} = [3, -2, a] \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = [3, -2, 6 - 6 + a] = [3, -2, a].
\]
We recognize the transformation $S$ as a rotation through an angle that is not an integer multiple of $\pi$.

As such there are no invariant lines and a unique invariant point. If the invariant point is $[x, y]$, it is determined by the equation:

\[
\begin{pmatrix}
\frac{3}{5} & -\frac{4}{5} & 2 \\
\frac{4}{5} & \frac{3}{5} & 6 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
1
\end{pmatrix}
= 
\begin{pmatrix}
x \\
y \\
1
\end{pmatrix}
\]

\[
x = \frac{3}{5}x - \frac{4}{5}y + 2, \quad y = \frac{4}{5}x + \frac{3}{5}y + 6,
\]

\[
5x = 3x - 4y + 10, \quad 5y = 4x + 3y + 30,
\]

\[
2x + 4y = 10, \quad -4x + 2y = 30,
\]

\[
x = 5 - 2y, \quad 2y - 4(5 - 2y) = 30,
\]

\[
10y = 50, \quad y = 5,
\]

\[
x = 5 - 2y = 5 - 10 = -5.
\]

So the rotation is centered at the point $(-5, 5)$ and the angle of rotation is $\arcsin\left(\frac{4}{5}\right) = 0.927295$ radians or 53.1301 degrees counter-clockwise.
Question 2

Let \( A = [3, 1], \ B = [5, 3], \ C = [5, -1], \ P = [-2, 1], \ Q = [0, -1] \) and \( R = [-4, -1] \) be points in the plane.
Plot the triangles \( ABC \) and \( PQR \) and describe a Euclidean transformation \( T \) that maps \( ABC \) to \( PQR \).
Also give a matrix representation of the transformation \( T \).
Does there exist a Euclidean transformation mapping \( ABC \) to \( PRQ \)?
Explain your answer.

Sketching the triangles, we first see that \( ABC \) is a right-angled isosceles triangle, with the base \( BC \) vertical of length 4 units.
We may verify this by computing the side lengths:

- \( |AB| = ||[5, 3] - [3, 1]| = ||2, 2|| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}. \)
- \( |AC| = ||[5, -1] - [3, 1]| = ||2, -2|| = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}. \)
- \( |BC| = ||[5, 3] - [5, -1]| = ||0, 4|| = \sqrt{0^2 + 4^2} = \sqrt{16} = 4. \)

Since \( |BC|^2 = |AC|^2 + |AB|^2 = 16 \), the triangle is right-angled at \( A \).
Since \( |AC| = |AB| \), the triangle is isosceles, with base angle forty-five degrees.

Similarly, we see that \( PQR \) is a right-angled isosceles triangle, with the base \( QR \) horizontal of length 4 units.
We may verify this by computing the side lengths:

- \( |PQ| = ||[-2, 1] - [0, -1]| = ||-2, 2|| = \sqrt{(-2)^2 + 2^2} = \sqrt{8} = 2\sqrt{2}. \)
- \( |PR| = ||[-2, 1] - [-4, -1]| = ||2, 2|| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}. \)
- \( |QR| = ||[0, -1] - [-4, -1]| = ||4, 0|| = \sqrt{4^2 + 0^2} = \sqrt{16} = 4. \)

Since \( |QR|^2 = |PQ|^2 + |PR|^2 = 16 \), the triangle is right-angled at \( R \).
Since \( |PQ| = |PR| \), the triangle is isosceles, with base angle forty-five degrees.
We move $ABC$ to $PQR$ in two steps:

- First we translate $A = [3, 1]$ to $P = [-2, 1]$, by moving five units to the left.
  The translation vector is $AP = P - A = [-2, 1] - [3, 1] = [-5, 0]$.
  The translation is $(x, y) \rightarrow (x - 5, y)$.
  The matrix for the translation is:

$$T_1 = \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

- Next we finish by rotating about $P$ through ninety degrees clockwise.
  The matrix for this is of the form:

$$T_2 = \begin{pmatrix} \cos(\frac{\pi}{2}) & \sin(\frac{\pi}{2}) & a \\ -\sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & a \\ -1 & 0 & b \\ 0 & 0 & 1 \end{pmatrix}.$$ 

The point $P$ must be the center of this rotation, which gives the matrix equation:

$$\begin{pmatrix} 0 & 1 & a \\ -1 & 0 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 + a \\ 2 + b \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix},$$

$$1 + a = -2, \quad a = -3, \quad 2 + b = 1, \quad b = -1.$$ 

- Finally we have $T = T_2T_1$:

$$T \cdot T_2T_1 = \begin{pmatrix} 0 & 1 & -3 \\ -1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -3 \\ -1 & 0 & 4 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

We may check by acting on the three points $A$, $B$ and $C$, simultaneously (written as column vectors):

$$T(ABC) = \begin{pmatrix} 0 & 1 & -3 \\ -1 & 0 & 4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 5 & 5 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 & -4 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix} = PQR.$$
Alternatively, once we have identified the rotational component as $-90$ degrees, we know that $T$ takes the form, for some $p$ and $q$:

$$
T = \begin{bmatrix}
\cos(\frac{\pi}{2}) & \sin(\frac{\pi}{2}) & p \\
-\sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) & q \\
0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
0 & 1 & p \\
-1 & 0 & q \\
0 & 0 & 1
\end{bmatrix}
$$

Then we apply this to $A$ and require that we get $P$, giving the matrix equation:

$$
\begin{bmatrix}
0 & 1 & p \\
-1 & 0 & q \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix},
$$

$$
\begin{bmatrix} 1 + p \\ -3 + q \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix},
$$

$1 + p = -2$, $p = -3$, $-3 + q = 1$, $q = 4$,

$$
T = \begin{bmatrix}
0 & 1 & -3 \\
-1 & 0 & 4 \\
0 & 0 & 1
\end{bmatrix}
$$

The center of the rotation $T$ is the solution $(x, y)$ of the equation:

$$
\begin{bmatrix}
0 & 1 & -3 \\
-1 & 0 & 4 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix},
$$

$$
y - 3 = x$, $-x + 4 = y$,

$$
-(y - 3) + 4 = y$, $2y = 7$,

$$
y = \frac{7}{2}$, $x = y - 3 = \frac{7}{2} - 3 = \frac{1}{2}$.

So the required transformation is a ninety degree clockwise rotation about the point $(\frac{1}{2}, \frac{7}{2})$. 
Finally we can map $ABC$ to $PQR$: we just follow the transformation $T$ by a reflection $U$ in the vertical line through $P$.

Then $U$ maps $(x, y)$ to $(-4 - x, y)$, so has the matrix:

$$U = \begin{pmatrix} -1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The required transformation $V$ is then:

$$V = UT = \begin{pmatrix} -1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & -3 \\ -1 & 0 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & 4 \\ 0 & 0 & 1 \end{pmatrix}.$$

We may check that this works:

$$V(ABC) = \begin{pmatrix} -1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & -3 \\ -1 & 0 & 4 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & 4 \\ 0 & 0 & 1 \end{pmatrix} = PRQ.$$

This transformation is a glide reflection, with axis the line joining the midpoint $\left(\frac{1}{2}, 1\right)$ of $AP$ to the midpoint of $\left(\frac{5}{2}, -1\right)$ of $CQ$.

This line has the equation $2x + 2y - 3 = 0$.

Check:

$$[2, 2, -3]V = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & 4 \\ 0 & 0 & 1 \end{pmatrix} = [2, 2, -3] = (-1)[2, 2, -3].$$
Question 3

Describe the symmetries of the following frieze pattern: \ldots\text{SSSSS}\ldots.

- There are translations through any integer multiple of the cell width.
- There are no reflections.
- There are no glide reflections.
- There are rotations each through an angle of 180 degrees, centered either at the midpoint of a letter of the pattern, or at a point midway between any two adjacent such midpoints.