Question 1

Consider the following axiom system for a geometry:

- There are exactly three points.
- Each line is on exactly two points.
- There are exactly four lines.

Describe with proof all the possible models.
Give axioms that distinguish between the various models.
Illustrate with pictures and for each picture give the associated incidence matrix.
Also, for each model sketch its dual model.

There are three point pairs, since there are three points.
Clearly the four lines can be distributed arbitrarily amongst the three point pairs.
So the models are described by partitions of the number four into three (unordered) parts.
We catalogue these models by their incidence matrices.
The points are \{A, B, C\}.
The lines are \{p, q, r, s\}.
• (4, 0, 0).
  All lines go through the same point-pair.
  \[
  \begin{array}{ccc}
  A & B & C \\
  p & 1 & 1 & 0 \\
  q & 1 & 1 & 0 \\
  r & 1 & 1 & 0 \\
  s & 1 & 1 & 0 \\
  \end{array}
  \]

• (3, 1, 0).
  All lines, except for one, go through the same point-pair.
  Let this point-pair be points A and B.
  Then the fourth line must go through one of these, so, wlog, through A:
  \[
  \begin{array}{ccc}
  A & B & C \\
  p & 1 & 1 & 0 \\
  q & 1 & 1 & 0 \\
  r & 1 & 1 & 0 \\
  s & 1 & 0 & 1 \\
  \end{array}
  \]

• (2, 2, 0).
  A pair of lines through each of two point-pairs.
  These point-pairs must have a point in common.
  Let this be point A.
  Then there are two lines through A and B and two through A and C.
  \[
  \begin{array}{ccc}
  A & B & C \\
  p & 1 & 1 & 0 \\
  q & 1 & 1 & 0 \\
  r & 1 & 1 & 0 \\
  s & 1 & 0 & 1 \\
  \end{array}
  \]

• (2, 1, 1).
  A pair of lines through one point-pair, wlog through A and B and one line through each of the other point-pairs the (A, C) pair and the (B, C) pair.
  \[
  \begin{array}{ccc}
  A & B & C \\
  p & 1 & 1 & 0 \\
  q & 1 & 1 & 0 \\
  r & 1 & 0 & 1 \\
  s & 0 & 1 & 1 \\
  \end{array}
  \]
Distinguishing axioms:

- (4, 0, 0): There exists a point on no lines.
- (3, 1, 0): There exists a point on exactly one line.
- (2, 2, 0): There exist two distinct points each of which lies on exactly two lines.
- (2, 1, 1): There exists a unique point which lies on exactly two lines.

The dual models obey the axiom scheme:

- There are exactly four points.
- Each point lies on exactly two lines.
- There are exactly three lines.
Question 2
Consider the $\mathbb{Z}_{11}$ affine geometry.

- Find the equation of the line $AB$ through the following points:
  \[ A = (2, 7) \text{ and } B = (5, 3). \]
  
  The slope is $m = \frac{3 - 7}{5 - 2} = -\frac{4}{3} = \frac{-18}{6} = 6$.
  
  By the point-slope form of the equation of a line, the line has the equation:
  \[ y - 3 = 6(x - 5) = 6x - 30, \quad y = 6x - 27 = 6x + 6. \]

- Give a parametrization of the line $AB$.
  The direction vector along the line is $V = B - A = (5, 3) - (2, 7) = (3, -4)$.
  
  So the line has the parametric equation:
  \[ X = A + tV = [2, 7] + t[3, -4] = [2 + 3t, 7 + 7t]. \]

- Find the other points of the line $AB$. For our parametrization, we get
  $A$ for the case $t = 0$ and $B$ for the case $t = 1$.

  The other points on the line are:
  
  - $t = 2$: [8, 10],
  - $t = 3$: [0, 6],
  - $t = 4$: [3, 2],
  - $t = 5$: [6, 9],
  - $t = 6$: [9, 5],
  - $t = 7$: [1, 1],
  - $t = 8$: [4, 8],
  - $t = 9$: [7, 4],
  - $t = 10$: [10, 0].

- Find the intersection point of the line $AB$ with the line with equation
  \( 2x - 4y = 1 \).
  
  We substitute $y = 6x + 6$ into this equation giving
  \[ 0 = 2x - 4(6x + 6) - 1 = 2x - 24x - 25 = -3. \]

  This is a contradiction, so the lines are parallel and do not meet (except at infinity).
Question 3
Consider the $\mathbb{Z}_5$ projective geometry.

- Find the equation of the line $\mathcal{L}$ through the following points:

$$C = (2, 0, 1) \text{ and } D = (2, 3, 2).$$

Using the cross-product the desired equation is given by the vanishing of the following determinant:

$$\det \begin{vmatrix} x & y & z \\ 2 & 0 & 1 \\ 2 & 3 & 2 \end{vmatrix} = x \det \begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} - y \det \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} + z \det \begin{vmatrix} 2 & 0 \\ 2 & 3 \end{vmatrix}$$

$$x(0 - 3) - y(4 - 2) + z(6 - 0) = -3x - 2y + 6z = 2x + 3y + z.$$ 

So $\mathcal{L}$ has the equation $2x + 3y + z = 0$.

It is easily checked that both the points $C$ and $D$ lie on the line, as required.

Alternatively we may first write the line in parametric form:

$$X = (x, y, z) = sC + tD = s(2, 0, 1) + t(2, 3, 2),$$

$$x = 2s + 2t, \quad y = 3t, \quad z = s + 2t.$$ 

We need to eliminate the parameters between these equations.

By the second of these equations, we get $2y = 6t = t$, so $t = 2y$.

Then by the third of the equations, we get $s = z - 2t = z - 4y = y + z$.

Inserting into the first equation, we get:

$$x = 2s + 2t = 2(y + z) + 2(2y) = 2y + 2z + 4y = 2z + y.$$ 

Bringing all terms to the left and doubling gives:

$$x - y - 2z = 0, \quad 2x - 2y - 4z = 0, \quad 2x + 3y + z = 0.$$ 

This gives the same line as before.
• Determine the other points of the line \( L \).
Putting \( z = 0 \), we find that \( 2x + 3y = 0 \), with solution the point \((1, 1, 0)\) at infinity on the line.
Otherwise, we may scale so that \( z = 1 \) and then \( 2x + 3y + 1 = 0 \).
Doubling gives \( 4x + y + 2 = 0 \), so \( y = x + 3 \).
Running through the five possibilities for \( x \) gives the points:
- \((0, 3, 1)\),
- \((1, 4, 1)\),
- \((2, 0, 1)\),
- \((3, 1, 1)\),
- \((4, 2, 1)\).
Of these the point \((2, 0, 1)\) is \( C \) and the point \((1, 4, 1) = (2, 8, 2) = (2, 3, 2)\) is \( D \).
So the other points of the line are \((1, 1, 0)\), \((0, 3, 1)\), \((3, 1, 1)\) and \((4, 2, 1)\).

• Find the point \( p \) of intersection of the line \( L \) with the line \( M \) with equation \( x + 2y + 2z = 0 \).
We use the determinant formalism: the line \( L \) has line co-ordinates \([2, 3, 1]\) and the lines \( M \) has line co-ordinates \([1, 2, 2]\), so their intersection point has the line equation:
\[
\begin{vmatrix}
p & q & r \\
2 & 3 & 1 \\
1 & 2 & 2 \\
\end{vmatrix}
= p \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} - q \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + r \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}
\]
\[
= p(6 - 2) - q(4 - 1) + r(4 - 3) = 4p + 2q + r.
\]
So the intersection point is \((4, 2, 1)\) and it is easily checked that this point lies on both lines as required.

Alternatively, we solve the equations \( x + 2y + 2z = 0 \) and \( 2x + 3y + z = 0 \) simultaneously.
Adding the two equations gives \( 3x + 3z = 0 \), so \( x = -z \).
Inserting this into the first equation gives \( -z + 2y + 2z = 0 \), so \( 2y + z = 0 \),
so \( 6y + 3z = 0 \), so \( y = -3z = 2z \).
Then the solution is \((x, y, z) = (-z, 2z, z) = z(-1, 2, 1) = z(4, 2, 1)\) (where \( z \neq 0 \)) in agreement with our previous solution.
• Determine the line co-ordinates of the point \( p \) with respect to the base points \( C \) and \( D \).

We need to solve the linear system:

\[
p = sC + tD,
\]

\[
(4, 2, 1) = s(2, 0, 1) + t(2, 3, 2)
\]

\[
4 = 2s + 2t, \quad 2 = 3t, \quad 1 = s + 2t.
\]

The middle of these equations gives:

\[
2 = 3t, \quad 4 = 6t = t, \quad t = 4.
\]

Substituting \( t = 4 \) into the third equation gives:

\[
1 = s + 2(4), \quad s = 1 - 8 = -7 = 3.
\]

With \( s = 3 \) and \( t = 4 \), we see that all the three equations are satisfied (incidentally verifying that the point \((4, 2, 1)\) does lie on the line \( CD \)).

So we may take as projective parameters \((s, t)\) any non-zero multiple of the pair \((3, 4)\).

The projective ratio is \( s : t = 3 : 4 = 8 : 4 = 2 : 1 \).
Question 4

Let the vertices of a square in the plane be labelled (in order around the square) by the integers mod 4.

- Interpret the transformation $n \rightarrow n+1 \mod 4$ as a symmetry operation $\mathcal{R}$ of the square, with a picture and give the name of the transformation.

  We label the vertices 0, 1, 2 and 3 going clockwise around the square. Then this transformation maps $0 \rightarrow 1$, $1 \rightarrow 2$, $2 \rightarrow 3$ and $3 \rightarrow 0$.

  So $\mathcal{R}$ is a clockwise rotation through 90 degrees.

- Interpret the transformation $n \rightarrow -n \mod 4$ as a symmetry operation $\mathcal{S}$ of the square, with a picture and give the name of the transformation.

  This transformation maps $0 \rightarrow 0$, $1 \rightarrow 3$, $2 \rightarrow 2$ and $3 \rightarrow 1$.

  So $\mathcal{S}$ is a reflection in the diagonal connecting 0 to 2.

- Give formulas and pictures for the transformations $\mathcal{RS}$ and $\mathcal{SR}$.

  1. $\mathcal{RS}$ maps as follows (first $\mathcal{S}$, then $\mathcal{R}$):

     $1 \rightarrow 3 \rightarrow 0$, $2 \rightarrow 2 \rightarrow 3$, $3 \rightarrow 1 \rightarrow 2$, $0 \rightarrow 0 \rightarrow 1$,

     $1 \rightarrow 0$, $2 \rightarrow 3$, $3 \rightarrow 2$, $0 \rightarrow 1$.

     This is a reflection in the line connecting the midpoint of the (0, 1) edge with the midpoint of the (2, 3) edge.

  2. $\mathcal{SR}$ maps as follows (first $\mathcal{R}$, then $\mathcal{S}$):

     $1 \rightarrow 2 \rightarrow 2$, $2 \rightarrow 3 \rightarrow 1$, $3 \rightarrow 0 \rightarrow 0$, $0 \rightarrow 1 \rightarrow 3$,

     $1 \rightarrow 2$, $2 \rightarrow 1$, $3 \rightarrow 0$, $0 \rightarrow 3$.

     This is a reflection in the line connecting the midpoint of the (1, 2) edge with the midpoint of the (3, 0) edge.

Note that $\mathcal{RS} \neq \mathcal{SR}$, so the group of symmetries is non-commutative.
Question 5

Describe the symmetries of each of the following frieze patterns $\mathcal{X}$ and $\mathcal{Y}$ (understood to go on forever, forwards and backwards):

- **Pattern $\mathcal{X}$:** 
  
  - There are translations through any integer multiple of the basic cell $<$.
  - There are reflections through the central axis.
  - Combining each of the translations with the reflection gives a glide reflection.
  - There are no other symmetry transformations.

- **Pattern $\mathcal{Y}$:** 
  
  - There are translations through any integer multiple of the basic cell-pair $<>$.
  - There are reflections through the central axis.
  - Combining each of the translations with the central axis reflection gives a glide reflection.
  - There are reflections through vertical axes through either the center of any $<>$ or any $> <$.
  - Combining each of the translations with the reflection gives a glide reflection.
  - There are rotations through 180 degrees about an axis the center of any $<>$ or any $> <$.
  - There are no other symmetry transformations.