Integrated Calculus I Quiz 1 Solutions 01/16/04

Name
Signature

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Question 1

Let \( f(x) = x^2 - 1 \), defined for all real \( x \) and \( g(x) = \frac{1}{x+1} \), defined for all real \( x \neq -1 \).

Compute the compositions: \((f \circ f)(x), (g \circ g)(x), (g \circ f)(x)\) and \((f \circ g)(x)\), giving appropriate domains for these compositions.

- We have:
  \[
  (f \circ f)(x) = f(f(x)) = (f(x))^2 - 1 = (x^2 - 1)^2 - 1 = x^4 - 2x^2.
  \]
  \((f \circ f)(x)\) is defined for all real \( x \).

- We have:
  \[
  (g \circ g)(x) = g(g(x)) = \frac{1}{g(x) + 1} = \frac{1}{1/x + 1} = \frac{x + 1}{x + 1} = \frac{x + 1}{x + 2}.
  \]
  \((g \circ g)(x)\) is defined for all real \( x \), such that \( x \neq -1 \) and \( g(x) + 1 \neq 0 \).
  But \( g(x) + 1 = 0 \) if \( \frac{1}{x+1} = -1 \), or \( 1 = -x-1 \), or \( x = -2 \).
  So \((g \circ g)(x)\) is defined provided \( x \neq -1 \) and \( x \neq -2 \).

- We have:
  \[
  (g \circ f)(x) = g(f(x)) = \frac{1}{f(x) + 1} = \frac{1}{x^2 - 1 + 1} = \frac{1}{x^2}.
  \]
  \((g \circ f)(x)\) is defined for all real \( x \), such that \( f(x) + 1 \neq 0 \).
  But \( f(x) + 1 = 0 \) if \( x^2 - 1 + 1 = 0 \), or \( x^2 = 0 \), or \( x = 0 \).
  So \((g \circ f)(x)\) is defined for all non-zero real numbers \( x \).

- We have:
  \[
  (f \circ g)(x) = f(g(x)) = (g(x))^2 - 1 = \frac{1}{(x+1)^2} - 1 = \frac{1 - (x+1)^2}{(x+1)^2} = -\frac{x(x+2)}{(x+1)^2}.
  \]
  \((f \circ g)(x)\) is defined whenever \( g(x) \) is defined, so whenever \( x \neq -1 \).


**Question 2**

Let \( f(x) = x^2 + x + 1 \), defined for all real \( x \).

- Compute the following limit:

\[
m = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}.
\]

We have \( f(2) = 2^2 + 2 + 1 = 7 \), so we get:

\[
m = \lim_{x \to 2} \frac{x^2 + x + 1 - 7}{x - 2} = \lim_{x \to 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 3)}{x - 2} = \lim_{x \to 2} (x + 3) = 5.
\]

- Give the geometrical interpretation of the limit \( m \).

The quantity \( m \) gives one definition of the slope of the tangent line to the curve \( y = x^2 + x + 1 \) at the point where \( x = 2 \) and \( y = 7 \).

- Hence find the equation of the tangent line to the curve \( y = x^2 + x + 1 \) at \( x = 2 \).

By the point-slope form of the equation of a line, since the tangent line has slope 5 and goes through the point \((2, 7)\), the required tangent line has the equation:

\[
y - 7 = 5(x - 2), \quad y = 7 + 5x - 10, \quad y = 5x - 3.
\]

- Sketch the curve and its tangent line.

See the Maple Solutions for this.
Question 3

Compute the following limits, or say why the limit does not exist:

- \( \lim_{x \to 4} \left( \frac{x^2 - 6x + 8}{x^2 - 16} \right) \).
  
  \[
  \lim_{x \to 4} \left( \frac{x^2 - 6x + 8}{x^2 - 16} \right) = \lim_{x \to 4} \left( \frac{(x - 2)(x - 4)}{(x - 4)(x + 4)} \right)
  = \lim_{x \to 4} \left( \frac{x - 2}{x + 4} \right)
  = \frac{4 - 2}{4 + 4} = \frac{1}{4}.
  \]

- \( \lim_{x \to 0} \left( \frac{\sqrt{1 + x^2} - \sqrt{1 - x^2}}{x^2} \right) \).
  
  \[
  \lim_{x \to 0} \left( \frac{\sqrt{1 + x^2} - \sqrt{1 - x^2}}{x^2} \right)
  = \lim_{x \to 0} \left( \frac{(\sqrt{1 + x^2} - \sqrt{1 - x^2})(\sqrt{1 + x^2} + \sqrt{1 - x^2})}{x^2(\sqrt{1 + x^2} + \sqrt{1 - x^2})} \right)
  = \lim_{x \to 0} \left( \frac{(1 + x^2) - (1 - x^2)}{x^2(\sqrt{1 + x^2} + \sqrt{1 - x^2})} \right)
  = \lim_{x \to 0} \left( \frac{2x^2}{x^2(\sqrt{1 + x^2} + \sqrt{1 - x^2})} \right)
  = \lim_{x \to 0} \left( \frac{2}{\sqrt{1 + x^2} + \sqrt{1 - x^2}} \right)
  = \frac{2}{\sqrt{1} + 0 + \sqrt{1} - 0} = 1.
  \]
\[ \lim_{x \to 0} \left( \frac{\sin(2x)}{1 - \cos(x)} \right) \]

\[ \lim_{x \to 0} \left( \frac{\sin(2x)}{1 - \cos(x)} \right) \]

\[ = \lim_{x \to 0} \left( \frac{\sin(2x)}{2x} \right) \left( \frac{2x}{1 - \cos(x)} \right) \]

\[ = \lim_{x \to 0} \left( \frac{2x}{1 - \cos(x)} \right) \]

\[ = \lim_{x \to 0} \left( \frac{2x(1 + \cos(x))}{(1 - \cos(x))(1 + \cos(x))} \right) \]

\[ = \lim_{x \to 0} \left( \frac{2x(1 + \cos(x))}{1 - \cos^2(x)} \right) \]

\[ = \lim_{x \to 0} \left( \frac{2x(1 + \cos(x))}{\sin^2(x)} \right) \]

\[ = \lim_{x \to 0} \left( \frac{2x(2)}{\sin^2(x)} \right) \]

\[ = 4 \lim_{x \to 0} \left( \frac{x}{x^2} \right) \left( \frac{x^2}{\sin^2(x)} \right) \]

\[ = 4 \lim_{x \to 0} \left( \frac{1}{x} \right) \left( \frac{x}{\sin(x)} \right)^2 \]

\[ = 4 \lim_{x \to 0} \left( \frac{1}{x} \right) \]

This limit does not exist, since \( \frac{1}{x} \) grows infinitely large in size as \( x \) approaches zero.

Here we used the trigonometric identity \( \sin^2(x) + \cos^2(x) = 1 \) and the limit: \( \lim_{x \to 0} \frac{\sin(x)}{x} = 1 \), proved in class.