1. [5 points] Find an equation of the line that passes through the points \((-3, 2)\) and \((1, 10)\)

*Solution:* \( m = \frac{10 - 2}{1 + 3} = 2, \ b = 2 - 2 \cdot (-3) = 8 \). Equation: \( y = 2x + 8 \).

2. Evaluate the limit

(a) [5 points] \( \lim_{x \to 0} \frac{5x - x^2}{3x + x^2} \)

*Solution:* \( \lim_{x \to 0} \frac{5x - x^2}{3x + x^2} = \lim_{x \to 0} \frac{x(5 - x)}{x(3 + x)} = \lim_{x \to 0} \frac{5 - x}{3 + x} = \frac{5}{3} \).

(b) [5 points] \( \lim_{x \to -2} \frac{x^2 - 4}{2 + x} \)

*Solution:* \( \lim_{x \to -2} \frac{x^2 - 4}{2 + x} = \lim_{x \to -2} \frac{(x - 2)(x + 2)}{x + 2} = \lim_{x \to -2} (x - 2) = -4 \).

3. Find the derivative of each function. You need not simplify the result.

(a) [5 points] \( f(x) = 6\sqrt{x^2} - \frac{6}{\sqrt{x}} \)

*Solution:* \( f(x) = 6x^{2/3} - 6x^{-1/2}, \ f'(x) = 4x^{-1/3} + 3x^{-3/2} \)

(b) [5 points] \( f(t) = \frac{t^2 + 2}{t - 1} \)

*Solution:* \( f'(t) = \frac{2t(t - 1) - t^2 - 2}{(t - 1)^2} = \frac{t^2 - 2t - 2}{(t - 1)^2} \)

(c) [5 points] \( g(x) = x^3 \sqrt{3 - x^3} \)

*Solution:* \( g'(x) = 3x^2 \sqrt{3 - x^3} - \frac{3x^5}{2\sqrt{3 - x^3}} \)

(d) [5 points] \( g(t) = e^{2t} \ln(1 - 7t^2) \)

*Solution:* \( g(t) = 2e^{2t} \ln(1 - 7t^2) - e^{2t} \cdot \frac{14t}{1 - 7t^2} \)
4. [10 points] Use the definition of derivative to find the derivative of $f(x) = x^2$. (No credit will be given when the definition is not used).

Solution: 

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} (2x + h) = 2x$$

5. (a) [5 points] Find $\frac{dy}{dx}$ if $x^2y - xy^3 = 2$

Solution: 

$$2xy + x^2\frac{dy}{dx} - y^3 - 3xy^2\frac{dy}{dx} = 0, \quad \frac{dy}{dx} = \frac{y^3 - 2xy}{x^2 - 3xy^2}.$$ 

(b) [5 points] Find the equation for the tangent line to the curve $x^2y - xy^3 = 2$ at the point $(2, 1)$.

Solution: 

$$m = \frac{\frac{3}{2} - 2 \cdot 1}{2 - \frac{3}{2} \cdot 1^2} = \frac{-3}{-2} = \frac{3}{2}.$$ 

Tangent line equation: $y = 1 + \frac{3}{2}(x - 2), \quad y = \frac{3}{2}x - 2.$

6. For the function $f(x) = \frac{2x}{x^2 + 9}$

(a) [5 points] find horizontal asymptotes

Solution: 

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{2x}{x^2 + 9} = 0.$$ 

(b) [5 points] make a sign diagram for the derivative

Solution: 

$$f'(x) = \frac{-2x^2 + 18}{(x^2 + 9)^2} = \frac{-2(x + 3)(x - 3)}{(x^2 + 9)^2},$$

$f'(x) > 0$ when $-3 < x < 3$; $f'(x) < 0$ when $x < -3$ or $x > 3$.

(c) [5 points] find all relative maximum and minimum values

Solution: Relative minimum is at $x = -3$, $f(-3) = -\frac{1}{3}$, relative maximum is at $x = 3$, $f(3) = \frac{1}{3}$. 

2
7. [10 points] Maximum Profit: A furniture store can sell 30 chairs per week at a price of $70 each. The manager estimates that for each $3 price reduction she can sell two more chairs per week. The chairs cost the store $7 each. If \( x \) stands for the number of $3 price reductions, find the price of the chairs and the quantity that maximize the profit.

Solution: Price is \( p(x) = 70 - 3x \), the quantity sold is \( q(x) = 30 + 2x \).

Revenue is \( R(x) = p(x) \cdot q(x) = (70 - 3x)(30 + 2x) = 2100 + 50x - 6x^2 \).

Cost is \( C(x) = 7q(x) = 210 + 14x \).

Profit is \( P(x) = R(x) - C(x) = 2100 - 210 + 36x - 6x^2 \).

We maximize the profit by finding its derivative and CNs. \( P'(x) = 36 - 12x \).

CNs: \( P''(x) \) is defined everywhere, \( P''(x) = 0 \) when \( x = 3 \) which is the only CNs of \( P(x) \).

\( P''(x) = -12 < 0 \). Hence profit has a relative maximum at \( x = 3 \). This maximum is absolute b/c the graph of the profit is a parabola opened down.

\( p(3) = 70 - 9 = 61 \), \( q(3) = 30 + 6 = 36 \).

Answer: the price of the chairs that maximize the profit is $61 and the quantity is 36.

8. [10 points] Two cars start moving from the same point. One travels east at 40 mi/h and the other travels north at 30 mi/h. At what rate is the distance between the cars increasing two hours later?

Solution: Let \( x(t) \) be the distance traveled by the first car and \( y(t) \) be the distance traveled by the second car; \( \frac{dx}{dt} = 40, \frac{dy}{dt} = 30 \).

The distance between cars is \( d(t) = \sqrt{x^2 + y^2} \).

\[
\frac{dd}{dt} = \frac{1}{2\sqrt{x^2 + y^2}} \left( 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \right) = \frac{1}{\sqrt{x^2 + y^2}} \left( x \frac{dx}{dt} + y \frac{dy}{dt} \right)
\]

After three hours \( x = 3 \cdot 40 \) mi, \( y = 3 \cdot 30 \) mi, \( d = \sqrt{3^2 \cdot 40^2 + 3^2 \cdot 30^2} = 3 \cdot 50 = 150 \) mi.

Then
\[
\frac{dd}{dt} = \frac{1}{3 \cdot 50} \cdot (3 \cdot 40 \cdot 40 + 3 \cdot 30 \cdot 30) = \frac{40 \cdot 40 + 30 \cdot 30}{50} = \frac{4 \cdot 40 + 3 \cdot 30}{5} = \frac{250}{5}
\]

\( \frac{dd}{dt} = 50 \) mi/h.

Solution: \( P e^{0.08t} = 3P, \quad e^{0.08t} = 3, \quad \ln e^{0.08t} = \ln 3, \quad \frac{8}{100} t = \ln 3, \quad t = \frac{100 \ln 3}{8}, \quad t = 12.5 \ln 3 \) years.

10. [10 points] Find the area under the curve \( y = x^2 e^{x^3} \) when \( 0 \leq x \leq 2 \). Leave answer in exact form.

Solution: The area is \( A = \int_{0}^{2} x^2 e^{x^3} \, dx \).

Substitution \( u = x^3, \, du = 3x^2 \, dx, \, u(0) = 0, \, u(2) = 8 \). Then

\[
A = \frac{1}{3} \int_{0}^{8} e^u \, du = \frac{1}{3} e^u \Big|_{0}^{8} = \frac{e^8 - 1}{3}
\]

11. For the demand function \( d(x) = 20 - 0.2x^2 \) and supply function \( s(x) = 0.6x^2 \) find

(a) [10 points] the market demand level (the positive value of \( x \) at which the demand function intersects the supply function).

Solution: \( 20 - 0.2x^2 = 0.6x^2, \, 0.8x^2 = 20, \, x^2 = 25, \, A = x = 5 \).

(b) [10 points] the consumer’s surplus at the market demand level found in part (a).

Solution: The market price is \( B = s(5) = 15 \). Consumer’s surplus is

\[
\int_{0}^{5} (d(x) - B) \, dx = \int_{0}^{5} (5 - 0.2x^2) \, dx = \left[ 5x - \frac{x^3}{15} \right]_{0}^{5} = 25 - \frac{25}{3} = \frac{50}{3} = 16 \frac{2}{3}
\]

12. [10 points] The population of a town is increasing at the rate of \( 6t e^{t/2} \) people per year, where \( t \) is the number of years from now. Find the average gain in population during the next six years. Leave your answer in exact form.

Solution: The average gain in population during the next six years is

\[
\frac{1}{6} \int_{0}^{6} 6t e^{t/2} \, dt = \int_{0}^{6} t e^{t/2} \, dt
\]
By parts: \( u = t, \ du = dt, \ dv = e^{t/2} \, dt, \ v = 2e^{t/2}. \)

\[
= 2t e^{t/2}\bigg|_0^6 - \int_0^6 2e^{t/2} \, dt = 12e^3 - 4e^{t/2}\bigg|_0^6 = 12e^3 - 4(e^3 - 1) = 8e^3 + 4
\]

13. For the function \( f(x, y) = e^{x-2y} \ln x \)

(a) [10 points] find the domain

Solution: domain is \( \{(x, y) | x > 0\} \)

(b) [10 points] find partials \( f_x \) and \( f_{yx} \).

Solution: \( f_x = e^{x-2y} \ln x + \frac{e^{x-2y}}{x} = e^{x-2y} \left( \ln x + \frac{1}{x} \right) \)

\( f_{yx} = f_{xy} = -2e^{x-2y} \left( \ln x + \frac{1}{x} \right) \).

14. [10 points] If a company’s profit function is

\[
P(x, y) = 2xy - 2x^2 - 3y^2 + 4x + 18y + 72 \text{ thousand dollars,}
\]

find how many of each unit \( x \) and \( y \) should be produced in order to maximize the profit.

Solution: \( P_x = 2y - 4x + 4 = 0 \Rightarrow y = 2x - 2, \ P_y = 2x - 6y + 18 = 0, \ x - 3y + 9 = 0, \ x - 6x + 6 + 9 = 0, \ -5x + 15 = 0 \Rightarrow x = 3, \ y = 4. \) CP is \((3, 4)\).

\( P_{xx} = -4, \ P_{yy} = -6, \ P_{xy} = 2, \ D = 24 - 4 = 20 > 0, \ P_{xx} < 0. \) Relative maximum.

So profit is maximized when \( x = 3, \ y = 4. \)

15. [10 points] Use Lagrange multipliers to find the maximum value of the function \( f(x, y) = xy \) subject to the constraint \( 3x + y = 12. \)

[Hint: Find CP. The maximum value of the function is attained at CP.]

Solution: \( F(x, y, \lambda) = xy + 3x\lambda + y\lambda - 12\lambda \)

\( F_x = y + 3\lambda = 0, \ F_y = x + \lambda = 0, \ F_\lambda = 3x + y - 12 = 0 \)

\( \lambda = -\frac{y}{3} = -x, \ y = 3x, \quad F_\lambda = 3x + 3x - 12 = 0, \ x = 2, \ y = 6. \)

CP is \((2, 6)\). The maximum value of the function is \( f(2, 6) = 12. \)