1. (20 points) Find a nonzero vector orthogonal to the plane containing the 3 points $P = (3, 5, -1)$, $Q = (0, 1, -1)$ and $R = (2, -3, 1)$. Also, find the area of the triangle formed by $P$, $Q$, and $R$.

$$\vec{u} = \overrightarrow{PQ} = (0, 1, 1) - (3, 5, -1) = \langle -3, -4, 0 \rangle$$
$$\vec{v} = \overrightarrow{PR} = (2, -3, 1) - (3, 5, -1) = \langle -1, -8, 2 \rangle$$

$$\vec{u} \times \vec{v}$$

is perpendicular to both $\vec{u}$ and $\vec{v}$.

$$\begin{vmatrix}
1 & 4 & 1 \\
-3 & -4 & 0 \\
-1 & -8 & 2
\end{vmatrix} = \hat{i}((-4)(2) - 0(-8)) - \hat{j}((-3)(2) - 0(-1)) + \hat{k}((-3)(-8) - 1(1)(-4))$$

$$= -8 \hat{i} + 6 \hat{j} + 20 \hat{k} = \langle -8, 6, 20 \rangle$$

or any nonzero multiple of this.

Area of triangle $= \frac{1}{2}$ (area of parallelogram)

$$\text{base} = |\vec{v}|$$

$$\text{height} = |\vec{u}| \sin \theta$$

$$\text{Area} = \frac{1}{2} |\vec{v}||\vec{u}| \sin \theta = \frac{1}{2} |\vec{v} \times \vec{u}| = \frac{1}{2} |\vec{u} \times \vec{v}|$$

$$= \sqrt{(-8)^2 + (6)^2 + (20)^2}$$

$$= \sqrt{64 + 36 + 400}$$

$$= \sqrt{500}$$

$$= 5 \sqrt{50}$$

Area of triangle $= \frac{1}{2} (10 \sqrt{5})$

$= 5 \sqrt{5}$
2. a) (10 points) Consider the space curve

\[ \mathbf{r}(t) = (2 \sin(t), t, 2 \cos(t)) \]

Find the equation of the tangent line at the point \((2, \frac{\pi}{2}, 0)\). This should be written as a vector-valued function: \( \mathbf{l}(t) = (a_1 t + b_1, a_2 t + b_2, a_3 t + b_3) \). Also, find the unit tangent vector \( \mathbf{T}(t) \) for any \( t \).

b) (10 points) Find the equation of the tangent plane to the surface \( z = f(x, y) \) at the point \((0, 1, -2)\) if \( f(x, y) = e^x (x - 2y) \).

\[ \mathbf{r}'(t) = \left< 2 \cos(t), 1, -2 \sin(t) \right> \]
\[ \mathbf{r}'(\frac{\pi}{2}) = \left< 2 \cos(\frac{\pi}{2}), 1, -2 \sin(\frac{\pi}{2}) \right> = \left< 0, 1, -2 \right> \]
\[ \mathbf{l}(t) = t \mathbf{r}'(\frac{\pi}{2}) + \left< 2, \frac{\pi}{2}, 0 \right> = \left< 2, t+\frac{\pi}{2}, -2t \right> \]

\[ \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\left< 2 \cos(t), 1, -2 \sin(t) \right>}{\sqrt{(2 \cos(t))^2 + 1^2 + (-2 \sin(t))^2}} = \frac{\left< 2 \cos(t), 1, -2 \sin(t) \right>}{\sqrt{4 \cos^2 t + 1 + 4 \sin^2 t}} = \frac{\left< 2 \cos(t), 1, -2 \sin(t) \right>}{\sqrt{5}} \]

b) \( f_x = e^x (x - 2y + 1) \).  \( f_x(0,1) = e^0 (0 - 2 + 1) = -1 \)
\( f_y = -2e^x \).  \( f_y(0,1) = -2e^0 = -2 \)

\[ z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \]
\[ z + 2 = -1(x - 0) - 2(y - 1) \]
\[ z + z = -x - 2y + z \]
\[ x + 2y + z = 0 \]
3. (20 points)

a) Find the following limit if it exists, or show that the limit does not exist:

\[
\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2 + y^8}.
\]

b) Suppose \( z \) depends on \( x \) and \( y \) (i.e., \( z = z(x, y) \)) and that both \( x \) and \( y \) are independent variables. Use implicit differentiation (or perhaps some other method) to find \( \frac{\partial z}{\partial y} \) if:

\[
\frac{1}{2} y^2 z + xy + xz = \ln(xy).
\]

---

a) try \( y = mx \);

\[
limit_{x \to 0} f(x, mx) = \lim_{x \to 0} \frac{x \cdot m^4 \cdot x^4}{x^2 + m^8 \cdot x^8} = \lim_{x \to 0} \frac{m^4 \cdot x^3}{1 + m^8 \cdot x^6} = 0, \forall m.
\]

This looks exactly like a homework problem.

where \( y^4 = x \) is a path to try.

\[
\lim_{y \to 0} f(y^4, y) = \lim_{y \to 0} \frac{y^8}{y^8 + y^8} = \frac{1}{2}.
\]

If \( \lim f(x, y) \) were to exist, then we'd get the same values above (independent of the path taken). \( \therefore \) the limit does not exist.

b) Implicit Diff:

\[
(yz + \frac{1}{2} y^2 \frac{\partial z}{\partial y}) + x + x \frac{\partial z}{\partial y} = \frac{1}{x} (x) \sqrt{\ln(xy) = \ln(x) + \ln(y)}
\]

\[
\Rightarrow \frac{\partial z}{\partial y} = \frac{1}{y} - x - yz
\]

\[
\Rightarrow \frac{\partial z}{\partial y} = \frac{1}{y} - x - yz = \frac{1 - xy - y^2 z}{\frac{1}{2} y^3 + xy}
\]

Chain Rule

\[
P(x, y, z) = \frac{1}{2} y^2 z + xy + xz - \ln(xy) = 0.
\]

\[
\frac{\partial P}{\partial y} = \frac{\partial P}{\partial x} = \frac{\partial P}{\partial z} = \frac{yz + x - \frac{1}{2} z}{\frac{1}{2} y^2 + x} \Rightarrow \frac{\partial z}{\partial y} = \frac{yz + x - \frac{1}{2} z}{\frac{1}{2} y^2 + x} (z \text{ independent})
\]

\[
\frac{\partial P}{\partial x} = \frac{1}{2} y^2 + x
\]

\[
\frac{\partial P}{\partial y} = \frac{1}{2} y^2 + x
\]

\( \therefore \) (what's written above)
4.

a) (6 points) Show that: \( u(x, t) = e^{-4t} \sin(2x) \) satisfies the partial differential equation (PDE):
\[
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \iff \frac{2u}{\partial t} = \frac{2^2 u}{\partial x^2}
\]
for all \( x \in \mathbb{R} \) and \( t \geq 0 \).

b) (8 points) Let \( n \in \mathbb{R} \). Does \( u(x, t) = e^{-n^2t} \sin(nx) \) also satisfy the differential equation? Show why or why not.

c) (6 points) Suppose we restrict \( x \) so \( x \in [0, 2\pi] \), and impose the condition that \( \frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(2\pi, t) = 0 \) for all \( t \). What does \( n \) have to be for \( u(x, t) = e^{-n^2t} \sin(nx) \) to be a solution to the PDE with these new conditions?

4. (a) \( \frac{\partial u}{\partial t} = -4e^{-4t} \sin(2x) \)

\[
\begin{align*}
\frac{\partial u}{\partial t} &= -4e^{-4t} \sin(2x) \\
\frac{\partial u}{\partial x} &= e^{-4t} \cos(2x) \\
\frac{\partial^2 u}{\partial x^2} &= e^{-4t} (-4 \sin(2x)) \\
\end{align*}
\]

Substitute into the PDE:

\[-4e^{-4t} \sin(2x) - (-4e^{-4t} \sin(2x)) = 0\]

\[-4e^{-4t} \sin(2x) + 4e^{-4t} \sin(2x) = 0 \text{ for all } t \geq 0, \text{ and all } x \in \mathbb{R}.
\]

(b) \( \frac{\partial u}{\partial t} = -n^2e^{-n^2t} \sin(nx) \)

\[
\begin{align*}
\frac{\partial u}{\partial t} &= -n^2e^{-n^2t} \sin(nx) \\
\frac{\partial u}{\partial x} &= ne^{-n^2t} \cos(nx) \\
\frac{\partial^2 u}{\partial x^2} &= -n^2e^{-n^2t} \sin(nx)
\end{align*}
\]

Substitute:

\[-n^2e^{-n^2t} \sin(nx) - (-n^2e^{-n^2t} \sin(nx)) = 0.\]

4. (c) \( \frac{\partial u}{\partial x} = ne^{-n^2t} \cos(nx) \)

\[
\begin{align*}
\frac{\partial u}{\partial x}(0, t) &= ne^{-n^2t} \cos(0) \\
&= ne^{-n^2t} \\
\frac{\partial u}{\partial x}(2\pi, t) &= ne^{-n^2t} \cos(2\pi n) = ne^{-n^2t} \\
\end{align*}
\]

\[\Rightarrow \cos(2\pi n) = 1.\]

This only happens for \( n = 0, \pm 1, \pm 2, \pm 3, \ldots \]

Natural Numbers \( \mathbb{N} \).
5. (20 points) Find the maximum and minimum values of \( f(x, y, z) = xyz \) subject to the 
constraint: \( 2x^2 + y^2 + 3z^2 = 3. \)

Use Lagrange Multipliers.

\[
\nabla f = \lambda \nabla g, \quad \nabla f = \langle yz, xz, xy \rangle,
\[
\nabla g = \langle 4x, 2y, 6z \rangle
\]

\[
yz = \lambda 4x, \quad xz = \lambda 2y, \quad xy = \lambda 6z
\]

Multiply by \( \lambda \neq 0 \) if \( \lambda = 0 \)

\[
yz = xz = xy = 0 \quad \Rightarrow \quad yz = xz = xy = 0
\]

either \( y = 0 \) or \( z = 0 \)

\( x = 0 \) or \( y = 0 \).

\[x = 0, \quad y = 0, \quad z = 0, \quad x = 0, \quad y = 0, \quad z = 0\]

2 of them must be 0.

\[
\begin{aligned}
(0, 0, \pm 1) \\
(0, \pm \sqrt{3}, 0) \\
(\pm \sqrt{\frac{2}{3}}, 0, 0)
\end{aligned}
\]

\[\lambda = 0\]

Test \( f \) on all these points

\[
f(0,0,\pm 1) = f(0,0,\pm \sqrt{3},0) = f(\pm \sqrt{\frac{2}{3}},0,0) = 0.
\]

\[
f(\pm \frac{1}{2}, \pm 1, \frac{1}{\sqrt{3}}) = \begin{cases}
\frac{1}{16}, & \text{if all positive, or exactly 2 negatives} \\
-\frac{1}{16}, & \text{if all negatives, or exactly 2 positives}.
\end{cases}
\]

8 possible points

\[f\] has absolute max at \( \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{13}} \right), \left( \frac{1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{13}} \right), \left( -\frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{13}} \right), \left( -\frac{1}{\sqrt{2}}, -1, \frac{1}{\sqrt{13}} \right), \left( \frac{1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{13}} \right), \left( -\frac{1}{\sqrt{2}}, 1, -\frac{1}{\sqrt{13}} \right), \left( -\frac{1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{13}} \right), \left( \frac{1}{\sqrt{2}}, 1, -\frac{1}{\sqrt{13}} \right) \]

\[f\] has absolute min at \( \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{13}} \right), \left( \frac{1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{13}} \right), \left( -\frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{13}} \right), \left( -\frac{1}{\sqrt{2}}, -1, \frac{1}{\sqrt{13}} \right), \left( \frac{1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{13}} \right), \left( -\frac{1}{\sqrt{2}}, 1, -\frac{1}{\sqrt{13}} \right), \left( -\frac{1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{13}} \right), \left( \frac{1}{\sqrt{2}}, 1, -\frac{1}{\sqrt{13}} \right) \]