1. Determine the average value of the function \( f(x) = x^3 \ln x \) on \( 1 \leq x \leq 10 \).

\[
A.V. = \frac{1}{9} \int_1^{10} x^3 \ln x \, dx \\
= \frac{1}{9} \left( \frac{1}{4} x^4 \ln x - \frac{1}{16} x^4 \right)_1^{10} \\
= \frac{1}{9} \left( \frac{10000}{4} \ln 10 - \frac{10000}{16} + \frac{1}{16} \right)
\]

2. Determine the area bounded by the \( x \)-axis and the curve \( y = \frac{\sqrt{1-x^2}}{x^2} \) on \( \frac{1}{2} \leq x \leq 1 \).

Set up triangle with hypotenuse \( c = 1 \), opposite side \( a = x \) and adjacent side \( b = \sqrt{1-x^2} \).

Then \( x = \sin \theta \), \( dx = \cos \theta \, d\theta \) and \( \sqrt{1-x^2} = \cos \theta \)

\[
\int \frac{\sqrt{1-x^2}}{x^2} \, dx = \int \frac{\cos \theta}{\sin^2 \theta} \cos \theta \, d\theta \\
= \int \cot^2 \theta \, d\theta \\
= \int (\csc^2 \theta - 1) \, d\theta \\
= -\cot \theta - \theta \\
= -\frac{\sqrt{1-x^2}}{x} - \arcsin x
\]

\[
\int_{1/2}^1 \frac{\sqrt{1-x^2}}{x^2} \, dx = \left( \frac{\sqrt{1-x^2}}{x} - \arcsin x \right)_{1/2}^1 = -\frac{\pi}{2} + \sqrt{3} + \frac{\pi}{6} = \sqrt{3} - \frac{\pi}{3}
\]
3. Determine the volume of the solid generated by rotating about the $x$-axis the region in the first quadrant bounded by the functions $f(x) = \sqrt{x} e^x$ and $y = e^x$ on $0 \leq x \leq 1$.

$$V = \pi \int_0^1 \left( (e^x)^2 - (\sqrt{x} e^x)^2 \right) dx$$
$$= \pi \int_0^1 (e^{2x} - xe^{2x}) dx$$
$$= \pi \left( \frac{1}{2} e^{2x} - \frac{1}{2} xe^{2x} + \frac{1}{4} e^{2x} \right)_0^1$$
$$= \pi \left( \frac{1}{4} e^2 - \frac{3}{4} \right)$$

4. Determine the volume of the solid generated by rotating about the $y$-axis the region bounded by the $y$-axis, the $x$-axis, $x = 1$ and the function $f(x) = \frac{3}{x^2 + 5x + 4}$.

$$V = 2\pi \int_0^1 \frac{3x}{x^2 + 5x + 4} dx$$
$$= 2\pi \int_0^1 \left( \frac{4}{x + 4} - \frac{1}{x + 1} \right) dx$$
$$= 2\pi \left( 4 \ln (x + 4) - \ln (x + 1) \right)_0^1$$
$$= 2\pi \left( 4 \ln (5) - \ln (2) - 4 \ln (4) \right)$$
5. Determine the work to empty a bowl filled with water (62.4 lb/ft³) out of the top if the bowl fits the rotated parabola \( y = x^2 - 1 \) for \( 1 \leq x \leq 2 \) in feet.

\[
W = 62.4\pi \int_0^3 \left(\sqrt{y+1}\right)^2 (3-y) \, dy
= 62.4\pi \int_0^1 (-y^2 + 2y + 3) \, dy
= 62.4\pi \left( -\frac{1}{3}y^3 + y^2 + 3y \right)_0^1 = 62.4(9)\pi
\]

6. Evaluate the integral if it converges. If it diverges, show the diverging limit.

(a) \( \int_2^\infty \frac{3}{(x-1)^2} \, dx \) \hspace{1cm} \text{Converges}

\[
\int_2^\infty \frac{3}{(x-1)^2} \, dx = -\frac{3}{x-1}\bigg|_{x=2}^{x=t\to\infty} = 0 - (-3) = 3
\]

(b) \( \int_1^2 \frac{x}{x^2 - 1} \, dx \) \hspace{1cm} \text{Diverges}

\[
\int_1^2 \frac{x}{x^2 - 1} \, dx = \frac{1}{2} \ln (x^2 - 1)\bigg|_{x=1}^{x=2} \to \infty \text{ as } x \to 1
\]

(c) \( \int_1^2 \frac{3x}{\sqrt{x^2 - 1}} \, dx \)

\[
\int_1^2 \frac{3x}{\sqrt{x^2 - 1}} \, dx = 3\sqrt{x^2 - 1}\bigg|_{x=1}^{x=2} = 3\sqrt{3} - 0 = 3\sqrt{3}
\]
7. Vector Fundamentals

(a) Determine the unit vector in the direction of the given vector:

\[ \vec{w} = 2\vec{i} - 5\vec{j} + 8\vec{k} \quad \hat{u}_{\vec{w}} = \left\langle \frac{2}{\sqrt{93}}, -\frac{5}{\sqrt{93}}, \frac{8}{\sqrt{93}} \right\rangle \]

(b) Determine the scalar projection of \( \vec{F} = \langle 4, 7 \rangle \) onto \( \vec{d} = \langle 9, 2 \rangle \).

\[ \text{comp}_{\vec{d}} \vec{F} = \frac{\vec{F} \cdot \vec{d}}{|\vec{d}|} = \frac{50}{\sqrt{85}} \]

(c) Determine the angle \( \theta \) (in degrees) between the vectors \( \vec{v} = \langle -1, 7, 0 \rangle \) and \( \vec{w} = \langle 3, 4, 5 \rangle \).

\[ \cos \theta = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}||\vec{w}|} = \frac{-3 + 28}{\sqrt{50}\sqrt{50}} = \frac{25}{50} = \frac{1}{2} \]

\[ \theta = \frac{\pi}{6} \]

(d) Determine the vector equation of the line passing through points \( P(-2, 4, 1) \) and \( Q(3, 3, 3) \).

Direction vector is \( \vec{v} = \langle 3 + 2, 3 - 4, 3 - 1 \rangle = \langle 5, -1, 2 \rangle \)
Line: \( \vec{r}(t) = \langle -2 + 5t, 4 - 1t, 1 + 2t \rangle \)

(e) Determine the volume of the parallelopiped formed by the vectors \( \vec{a} = \langle 2, -4, 1 \rangle \), \( \vec{b} = \langle 5, -1, 4 \rangle \) and \( \vec{c} = \langle 1, 3, 8 \rangle \).

\[ \det \begin{vmatrix} 2 & -4 & 1 \\ 5 & -1 & 4 \\ 1 & 3 & 8 \end{vmatrix} = |2(8 - 12) - (-4)(40 - 4) + 1(15 + 1)| = 120 \]
8. For what value(s) of $a$ are the vectors $\langle a^2, -1, 3 \rangle$ and $\langle 2, a, -5 \rangle$ orthogonal (perpendicular)?

\[ \vec{v} \perp \vec{w} \iff \vec{v} \cdot \vec{w} = 0. \]

Therefore:

\[ 2a^2 - a - 15 = 0 \]

and

\[ (2a + 5)(a - 3) = 0 \quad \Rightarrow \quad a = \frac{5}{3} \text{ or } a = 3 \]

9. Determine the area of the parallelogram formed by the vectors $\vec{v} = \langle 1, 1, 4 \rangle$ and $\vec{w} = \langle -2, 3, 2 \rangle$

\[ \vec{n} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 4 \\ -2 & 3 & 2 \end{pmatrix} = -10\vec{i} - 10\vec{j} + 5\vec{k} \]

Area = $|\vec{n}| = \sqrt{225} = 15$.

10. Determine the equation of the plane contains the vectors $\vec{v} = \langle 4, 2, -1 \rangle$ and $\vec{w} = \langle 1, 3, -3 \rangle$ and contains the point $P(2, 3, -5)$.

\[ \vec{n} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 2 & -1 \\ 1 & 3 & -3 \end{pmatrix} = \langle -3, 11, 10 \rangle \]

\[ \langle -3, 10, 11 \rangle \cdot \langle x - 2, y - 3, z + 5 \rangle = 0 \quad \text{implies} \]

Plane: $-3x + 11y + 10 = -23$