Lecture 5

Duality
Resource Allocation

Recall the resource allocation problem \((m = 2, n = 3)\):

\[
\begin{align*}
\text{maximize} & \quad c_1 x_1 + c_2 x_2 + c_3 x_3 \\
\text{subject to} & \quad a_{11} x_1 + a_{12} x_2 + a_{13} x_3 \leq b_1 \\
& \quad a_{21} x_1 + a_{22} x_2 + a_{23} x_3 \leq b_2 \\
& \quad x_1, x_2, x_3 \geq 0,
\end{align*}
\]

where

\[
\begin{align*}
c_j &= \text{profit per unit of product } j \text{ produced} \\
b_i &= \text{units of raw material } i \text{ on hand} \\
a_{ij} &= \text{units of raw material } i \text{ required to produce one unit of product } j.
\end{align*}
\]

Forgoing production and selling off raw materials.

If we produce one unit less of product \(j\), then we free up:

- \(a_{1j}\) units of raw material 1 and
- \(a_{2j}\) units of raw material 2.

Selling these unused raw materials for \(y_1\) and \(y_2\) dollars/unit, yields \(a_{1j} y_1 + a_{2j} y_2\) dollars.

Only interested if this exceeds lost profit on product \(j\):

\[
a_{1j} y_1 + a_{2j} y_2 \geq c_j.
\]

We want these inequalities for \(j = 1, 2, 3\).

Consider a buyer offering to purchase our entire inventory. Subject to above constraints, buyer wants to minimize cost:

\[
\begin{align*}
\text{minimize} & \quad b_1 y_1 + b_2 y_2 \\
\text{subject to} & \quad a_{11} y_1 + a_{21} y_2 \geq c_1 \\
& \quad a_{12} y_1 + a_{22} y_2 \geq c_2 \\
& \quad a_{13} y_1 + a_{23} y_2 \geq c_3 \\
& \quad y_1, y_2 \geq 0.
\end{align*}
\]
Duality

Every Problem:

\[
\begin{align*}
\text{maximize} & \quad \sum_{j=1}^{n} c_j x_j \\
\text{subject to} & \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i & i = 1, 2, \ldots, m \\
& \quad x_j \geq 0 & j = 1, 2, \ldots, n,
\end{align*}
\]

Has a Dual:

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{m} b_i y_i \\
\text{subject to} & \quad \sum_{i=1}^{m} y_i a_{ij} \geq c_j & j = 1, 2, \ldots, n \\
& \quad y_i \geq 0 & i = 1, 2, \ldots, m.
\end{align*}
\]

Notes:

- Original problem is the primal problem.
- A problem is defined by its data (notation used for the variables is arbitrary).
- Dual is negative transpose of primal (see below).
- Dual of dual is primal.

Dual in “Standard” Form:

\[
\begin{align*}
\text{maximize} & \quad \sum_{i=1}^{m} -b_i y_i \\
\text{subject to} & \quad \sum_{i=1}^{m} -a_{ij} y_i \leq -c_j & j = 1, 2, \ldots, n \\
& \quad y_i \geq 0 & i = 1, 2, \ldots, m.
\end{align*}
\]
Weak Duality Theorem

If \((x_1, x_2, \ldots, x_n)\) is feasible for the primal and \((y_1, y_2, \ldots, y_m)\) is feasible for the dual, then

\[
\sum_j c_j x_j \leq \sum_i b_i y_i.
\]

Why? Consider the following chain of inequalities:

\[
\begin{align*}
\sum_j c_j x_j &\leq \sum_j \left( \sum_i y_i a_{ij} \right) x_j \\
&= \sum_{ij} y_i a_{ij} x_j \\
&= \sum_i \left( \sum_j a_{ij} x_j \right) y_i \\
&\leq \sum_i b_i y_i,
\end{align*}
\]

An important question:

Is there a gap between the largest primal value and the smallest dual value?
Answer: Later (Strong Duality Theorem)
Simplex Method and Duality

An Example:

Its Dual:

Notes:

- Dual is negative transpose of primal.
- Primal is feasible, dual is not.

Use primal to choose pivot: $x_2$ enters, $w_2$ leaves.
Make analogous pivot in dual: $z_2$ leaves, $y_2$ enters.
**Second Iteration**

After First Pivot:

Primal (still feasible):

\[
\begin{array}{cccccc}
\text{obj} & = & \frac{3}{2} & + & \frac{-3}{2} & x_1 + \frac{-1}{2} & v_2 + \frac{1}{2} & x_3 \\
v_1 & = & \frac{3}{4} & - & \frac{3}{4} & x_1 - \frac{1}{4} & v_2 - \frac{9}{4} & x_3 \\
x_2 & = & \frac{3}{4} & - & \frac{3}{4} & x_1 - \frac{1}{4} & v_2 - \frac{1}{4} & x_3 \\
\end{array}
\]

Dual (still not feasible):

\[
\begin{array}{cccccc}
\text{obj} & = & \frac{-3}{2} & + & \frac{-3}{4} & y_1 + \frac{-3}{4} & z_2 \\
v_1 & = & \frac{3}{2} & - & \frac{3}{4} & y_1 - \frac{3}{4} & z_2 \\
y_2 & = & \frac{1}{2} & - & \frac{1}{4} & y_1 - \frac{1}{4} & z_2 \\
z_3 & = & \frac{-1}{2} & - & \frac{-9}{4} & y_1 - \frac{-1}{4} & z_2 \\
\end{array}
\]

Note: negative transpose property intact.

Again, use primal to pick pivot: \(x_3\) enters, \(w_1\) leaves. Make analogous pivot in dual: \(z_3\) leaves, \(y_1\) enters.
**After Second Iteration**

**Primal:**

<table>
<thead>
<tr>
<th></th>
<th>5/3</th>
<th>-4/3</th>
<th>x1</th>
<th>-5/9</th>
<th>v2</th>
<th>-2/3</th>
<th>v1</th>
</tr>
</thead>
<tbody>
<tr>
<td>x3</td>
<td>1/3</td>
<td>-1/3</td>
<td>x1</td>
<td>1/9</td>
<td>x1</td>
<td>-2/3</td>
<td>v1</td>
</tr>
<tr>
<td>x2</td>
<td>2/3</td>
<td>-2/3</td>
<td>x1</td>
<td>2/9</td>
<td>x1</td>
<td>-1/9</td>
<td>v1</td>
</tr>
</tbody>
</table>

**Dual:**

<table>
<thead>
<tr>
<th></th>
<th>-5/3</th>
<th>-1/3</th>
<th>z3</th>
<th>-2/3</th>
<th>z2</th>
</tr>
</thead>
<tbody>
<tr>
<td>z1</td>
<td>4/3</td>
<td>z3</td>
<td>z3</td>
<td>z3</td>
<td>z2</td>
</tr>
<tr>
<td>y2</td>
<td>5/9</td>
<td>-1/9</td>
<td>z3</td>
<td>-2/9</td>
<td>z2</td>
</tr>
<tr>
<td>y1</td>
<td>2/9</td>
<td>-4/9</td>
<td>z3</td>
<td>1/9</td>
<td>z2</td>
</tr>
</tbody>
</table>

**Notes:**

- Primal is **optimal**.
- Negative transpose property is intact.
- Dual is **optimal**.

**Conclusion**

Simplex method applied to primal problem (two phases, if necessary), solves both the primal and the dual.
**Strong Duality Theorem**

Conclusion on previous slide is the essence of the strong duality theorem which we now state:

If the primal problem has an optimal solution, 
\[ x^* = (x_1^*, x_2^*, \ldots, x_n^*) , \]
then the dual also has an optimal solution, 
\[ y^* = (y_1^*, y_2^*, \ldots, y_m^*) , \]
such that 
\[ \sum_{j} c_j x_j^* = \sum_{i} b_i y_i^* . \]

**Paraphrase:**

If primal has an optimal solution, then there is no duality gap.
Duality Gap

Four possibilities:

- Primal optimal, dual optimal (no gap).
- Primal unbounded, dual infeasible (no gap).
- Primal infeasible, dual unbounded (no gap).
- Primal infeasible, dual infeasible (infinite gap).

Example of infinite gap:

\[
\begin{align*}
\text{maximize} & \quad 2x_1 - x_2 \\
\text{subject to} & \quad x_1 - x_2 \leq 1 \\
& \quad -x_1 + x_2 \leq -2 \\
& \quad x_1, x_2 \geq 0.
\end{align*}
\]
**Complementary Slackness**

At optimality, we have

\[ x_j z_j = 0, \quad \text{for } j = 1, 2, \ldots, n, \]
\[ w_i y_i = 0, \quad \text{for } i = 1, 2, \ldots, m. \]

Why? Recall the proof of the Weak Duality Theorem:

\[
\sum_j c_j x_j \leq \sum_j (c_j + z_j) x_j
\]

\[
= \sum_j \left( \sum_i y_i a_{ij} \right) x_j
\]

\[
\Rightarrow \quad \sum_{ij} y_i a_{ij} x_j
\]

\[
= \sum_i \left( \sum_j a_{ij} x_j \right) y_i
\]

\[
= \sum_i (b_i - w_i) y_i
\]

\[
\leq \sum_i b_i y_i.
\]

The inequalities come from the fact that

\[ x_j z_j \geq 0, \quad \text{for } j = 1, 2, \ldots, n, \]
\[ w_i y_i \geq 0, \quad \text{for } i = 1, 2, \ldots, m. \]

By Strong Duality Theorem, the inequalities are equalities at optimality.
**Dual Simplex Method**

When: dual feasible, primal infeasible (i.e., pinks on the left, not on top).

**An Example.** Showing both primal and dual dictionaries:

Looking at dual dictionary: $y_2$ enters, $z_2$ leaves.
On the primal dictionary: $w_2$ leaves, $x_2$ enters.

After pivot:
**Dual Simplex Method: Second Pivot**

Going in, we have:

Looking at dual:
- \( y_1 \) enters, \( z_4 \) leaves.

Looking at primal:
- \( w_1 \) leaves, \( x_4 \) enters.

**Dual Simplex Method Pivot Rule:**
Refering to the primal dictionary,

- Pick leaving variable from those rows that are infeasible.
- Pick entering variable from a box with a negative value and which can be increased the least (on the dual side).

Next primal dictionary shown on next page...
## Dual Simplex Method: Third Pivot

Going in, we have:

<table>
<thead>
<tr>
<th></th>
<th>obj</th>
<th>$x_1$</th>
<th>$v_2$</th>
<th>$x_3$</th>
<th>$v_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$-15.7143$</td>
<td>$-5.1429$</td>
<td>$-2.2857$</td>
<td>$0.0$</td>
<td>$-1.4286$</td>
</tr>
<tr>
<td>$x_2$</td>
<td>$2.7143$</td>
<td>$-0.1429$</td>
<td>$-0.2857$</td>
<td>$0.0$</td>
<td>$-0.4286$</td>
</tr>
<tr>
<td>$w_3$</td>
<td>$-0.1429$</td>
<td>$-0.5714$</td>
<td>$-0.1429$</td>
<td>$0.0$</td>
<td>$0.2857$</td>
</tr>
</tbody>
</table>

Which variable must leave and which must enter?

See next page...
**Dual Simplex Method: Third Pivot—Answer**

Answer is: \(x_2\) leaves, \(x_1\) enters.

Resulting dictionary is OPTIMAL:

<table>
<thead>
<tr>
<th></th>
<th>(x_4)</th>
<th>(x_1)</th>
<th>(x_3)</th>
<th>(w_3)</th>
<th>(w_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>obj</strong></td>
<td>17.0</td>
<td>-9.0</td>
<td>-1.0</td>
<td>0.0</td>
<td>-4.0</td>
</tr>
<tr>
<td><strong>x_4</strong></td>
<td>2.75</td>
<td>-0.25</td>
<td>-0.25</td>
<td>0.0</td>
<td>-0.5</td>
</tr>
<tr>
<td><strong>x_1</strong></td>
<td>0.25</td>
<td>-1.75</td>
<td>0.25</td>
<td>0.0</td>
<td>-0.5</td>
</tr>
<tr>
<td><strong>w_3</strong></td>
<td>2.0</td>
<td>7.0</td>
<td>0.0</td>
<td>3.0</td>
<td>2.0</td>
</tr>
</tbody>
</table>
Dual-Based Phase I Method

An Example:

Notes:

- Two objective functions: the true objective (on top), and a fake one (below it).
- For Phase I, use the fake objective—it’s dual feasible.
- Two right-hand sides: the real one (on the left) and a fake (on the right).
- Ignore the fake right-hand side—we’ll use it in another algorithm later.

Phase I—First Pivot: $w_3$ leaves, $x_1$ enters.

After first pivot:
## Dual-Based Phase I Method—Second Pivot

Recall current dictionary:

<table>
<thead>
<tr>
<th></th>
<th>obj</th>
<th>w1</th>
<th>w2</th>
<th>x1</th>
<th>w4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-12.0</td>
<td>-6.0</td>
<td>-9.0</td>
<td>3.0</td>
<td>5.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>3.0</td>
<td>4.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>-1.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>-1.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>-2.0</td>
<td></td>
</tr>
</tbody>
</table>

Dual pivot: $w_2$ leaves, $x_2$ enters.

After pivot:

<table>
<thead>
<tr>
<th></th>
<th>obj</th>
<th>w1</th>
<th>w2</th>
<th>x1</th>
<th>w4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-3.0</td>
<td>-1.5</td>
<td>1.5</td>
<td>1.5</td>
<td>2.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.0</td>
<td>-0.6667</td>
<td>-0.3333</td>
<td>0.3333</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>-0.5</td>
<td></td>
<td>-0.5</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>-1.0</td>
<td></td>
</tr>
</tbody>
</table>
Dual-Based Phase I Method—Third Pivot

Current dictionary:

| obj  | -3.0 | + | -1.0 | w3 | + | 1.0 | w2 | + | -2.0 | x3 |
|------|------|+|------|----|+|------|----|+|-------|----|
| w1   | -1.5 | + | 1.0  | - | 0.5 | w3 | - | 0.5  | w2 | - | 5.5  | x3 |
| x2   | 1.5  | + | -0.6667 | - | 0.5 | w3 | - | 0.1667 | w2 | - | 0.1667 | x3 |
| x1   | 1.5  | + | -0.3333 | - | 0.5 | w3 | - | 0.1667 | w2 | - | -1.1667 | x3 |
| w4   | 2.0  | + | 0.3333 | - | 0.5 | w3 | - | 0.3333 | w2 | - | -2.3333 | x3 |

Dual pivot: $w_1$ leaves, $w_2$ enters.

After pivot:

| obj  | 0.0  | + | 0.0  | w3 | + | 2.0  | w1 | + | 9.0   | x3 |
|------|------|+|------|----|+|------|----|+|-------|----|
| w2   | 3.0  | + | -2.0 | - | 1.0 | w3 | - | -2.0 | w1 | - | -11.0 | x3 |
| x2   | 2.0  | + | -1.0 | - | 0.6667 | w3 | - | -0.3333 | w1 | - | -1.6667 | x3 |
| x1   | 1.0  | + | 0.0  | - | -0.3333 | w3 | - | 0.3333 | w1 | - | 0.6667 | x3 |
| w4   | 1.0  | + | 1.0  | - | -0.6667 | w3 | - | 0.6667 | w1 | - | 1.3333 | x3 |

It’s feasible!
Fourth Pivot—Phase II

Current dictionary:

<table>
<thead>
<tr>
<th></th>
<th>0.0</th>
<th>+</th>
<th>0.0</th>
<th>v3</th>
<th>+</th>
<th>2.0</th>
<th>w1</th>
<th>+</th>
<th>9.0</th>
<th>x3</th>
</tr>
</thead>
<tbody>
<tr>
<td>w2</td>
<td>3.0</td>
<td>+</td>
<td>-2.0</td>
<td></td>
<td>-1.0</td>
<td>v3</td>
<td>-</td>
<td>-2.0</td>
<td>v1</td>
<td>-</td>
</tr>
<tr>
<td>x2</td>
<td>2.0</td>
<td>+</td>
<td>-1.0</td>
<td></td>
<td>-0.6667</td>
<td>v3</td>
<td>-</td>
<td>-0.3333</td>
<td>v1</td>
<td>-</td>
</tr>
<tr>
<td>x1</td>
<td>1.0</td>
<td>+</td>
<td>0.0</td>
<td></td>
<td>-0.3333</td>
<td>v3</td>
<td>-</td>
<td>0.3333</td>
<td>v1</td>
<td>-</td>
</tr>
<tr>
<td>w4</td>
<td>1.0</td>
<td>+</td>
<td>1.0</td>
<td></td>
<td>-0.6667</td>
<td>v3</td>
<td>-</td>
<td>0.6667</td>
<td>v1</td>
<td>-</td>
</tr>
</tbody>
</table>

It’s feasible.
Ignore fake objective.
Use the real thing (top row).

Primal pivot: $x_3$ enters, $w_4$ leaves.

After pivot:

<table>
<thead>
<tr>
<th></th>
<th>6.75</th>
<th>+</th>
<th>4.5</th>
<th>v3</th>
<th>+</th>
<th>-2.5</th>
<th>w1</th>
<th>+</th>
<th>-6.75</th>
<th>v4</th>
</tr>
</thead>
<tbody>
<tr>
<td>w2</td>
<td>11.25</td>
<td>+</td>
<td>6.25</td>
<td></td>
<td>-6.5</td>
<td>v3</td>
<td>-</td>
<td>3.5</td>
<td>v1</td>
<td>-</td>
</tr>
<tr>
<td>x2</td>
<td>3.25</td>
<td>+</td>
<td>0.25</td>
<td></td>
<td>-1.5</td>
<td>v3</td>
<td>-</td>
<td>0.5</td>
<td>v1</td>
<td>-</td>
</tr>
<tr>
<td>x1</td>
<td>0.5</td>
<td>+</td>
<td>-0.5</td>
<td></td>
<td>0.0</td>
<td>v3</td>
<td>-</td>
<td>0.0</td>
<td>v1</td>
<td>-</td>
</tr>
<tr>
<td>x3</td>
<td>0.75</td>
<td>+</td>
<td>0.75</td>
<td></td>
<td>-0.5</td>
<td>v3</td>
<td>-</td>
<td>0.5</td>
<td>v1</td>
<td>-</td>
</tr>
</tbody>
</table>

Problem is **unbounded!**