Lecture 21

More Applications

Options Pricing
Farkas’ Lemma for Systems in Equality Form

Recall Farkas’ Lemma:

Lemma. The system $Ax \leq b$ has no solutions if and only if there is a $y$ such that

$$A^T y = 0$$
$$y \geq 0$$
$$b^T y < 0.$$

Today we need it in another form:

Lemma. The system $Ax = b, x \geq 0$ has no solutions if and only if there is a $y$ such that

$$A^T y \geq 0$$
$$b^T y < 0.$$

Proof is completely analogous to the one we had before. Hence, omitted.
Consider a collection of \( n \) assets (possible investments).

Suppose that one time period will result in one of \( m \) possible scenarios of outcomes.

Let

\[
    r_{ij} = \text{return from asset } j \text{ under scenario } i
\]

and

\[
    R = \begin{bmatrix} r_{ij} \end{bmatrix}.
\]

Note: these returns are in dollars-returned per item-of-investment as opposed to our Markowitz model in which returns were measured in dollars-returned per dollar-invested.

**Problem:** Determine a consistent set of prices for the investments:

\[
    p_j = \text{price (in dollars) for asset } j.
\]
Arbitrage

**Big Assumption 1:** We can hold positive or negative quantities of each asset—the return is the same.

Never satisfied in practice. If I give a bank 1 dollar to hold, they will return it after a year with 4% interest but if I give a bank $−1$ dollar to hold (i.e., I borrow a dollar), they will give $−1$ back to me with 10% interest.

It is, however, generally assumed to be true, at least for the big players.

Let

$$x_j = \text{number of units of asset } j \text{ I hold.}$$

Wealth at end of time period under scenario $i$:

$$w_i = \sum_j r_{ij} x_j.$$  

In matrix notation:

$$w = Rx.$$  

Recall: the total current “price” for this portfolio is:

$$p^T x$$

An arbitrage is a portfolio $x$ which is guaranteed (under every scenario) to have nonnegative value at the end of the time period but which has a negative price at the beginning:

$$Rx \geq 0 \quad \text{and} \quad p^T x < 0.$$  

**Big Assumption 2:** The scenarios considered cover all possibilities.
Efficient Market Assumption

**Assumption:** Prices will equilibrate so as to eliminate arbitrage.

**Theorem.** There is no arbitrage if and only if there is a vector $y$ that satisfies:

$$R^T y = p$$

$$y \geq 0.$$

**Proof.** Immediate from Farkas’ Lemma ($A = R^T$, $b = p$, and $x$ and $y$ interchanged).

**Notes.**

- If $m = n$ and $R$ is nonsingular, then the equality constraints uniquely determine $y$. Then, only need to check nonnegativity.
- If $p$ is arbitrage-free, then any nonnegative constant times $p$ is too. Therefore, at least one of the prices in $p$ needs to be fixed arbitrarily (by, e.g., Alan Greenspan).
**Options**

**Definition.** An option is a contract giving one the “option” to buy a specific stock at a specific price at a specific time in the future.

The price, usually denote $K$, is called the **strike price**.

Consider a single-time-period market consisting of

- A Stock
- A Bond
- An Option on the Stock

Let $\tilde{S}$ denote the value of the stock at the end of the time period.

If $\tilde{S} > K$, then the option holder will exercise the option by buying the stock at $K$ dollars and immediately selling it for $\tilde{S}$ dollars, yielding a profit of $\tilde{S} - K$ dollars.

If $\tilde{S} \leq K$, then the option holder will let the option expire and so at the end the value to the holder is zero dollars.

To summarize: the value at the end of the time period is

$$\max(0, \tilde{S} - K).$$

The fundamental question is: how much should one pay for such an option?
Options Pricing

Suppose that there are only two scenarios:

- The stock goes up by a factor $u > 1$, or
- down by a factor $d < 1$.

Under both scenarios, the bond goes up by a factor $r > 1$.

Suppose at the beginning that the stock price is $S$, the price of the bond is $B$, and of course the price of the option is to be determined. Let’s denote it by $O$.

The matrix $R$ is then given by:

$$ R = \begin{bmatrix} Su & Sd \\ Br & Br \\ \text{max}(0, Su - K) & \text{max}(0, Sd - K) \end{bmatrix} $$

and the vector $p$ is given by:

$$ p = \begin{bmatrix} S \\ B \\ O \end{bmatrix} $$

The no-arbitrage theorem says there must exist a vector $y = [y_u \ y_d]^T$ such that

$$ \begin{bmatrix} Su & Sd \\ Br & Br \\ \text{max}(0, Su - K) & \text{max}(0, Sd - K) \end{bmatrix} \begin{bmatrix} y_u \\ y_d \end{bmatrix} = \begin{bmatrix} S \\ B \\ O \end{bmatrix} $$

$$ \begin{bmatrix} y_u \\ y_d \end{bmatrix} \geq 0 $$
Black Scholes Formula

The first two equations can be solved for $y_u$ and $y_d$:

$$
\begin{bmatrix}
y_u \\
y_d
\end{bmatrix} = \begin{bmatrix}
Su & Sd \\
Br & Br
\end{bmatrix}^{-1} \begin{bmatrix}
S \\
B
\end{bmatrix} = \frac{1}{r(u-d)} \begin{bmatrix}
r - d \\
u - r
\end{bmatrix}
$$

Then the last equation can be solved for $O$:

$$
O = y_u \max(0, Su - K) + y_d \max(0, Sd - K)
$$

This option pricing formula is the discrete analogue of the famous Black-Scholes formula.

Note:

- The nonnegativity requirement on $y$ forces us to assume that $d < r < u$. 
Suppose that one believes the up-scenario will happen with probability $\alpha$ and the down-scenario will happen with probability $\beta = 1 - \alpha$.

Then, one possible formula for the option price would be the expected present value of the option:

$$E_{\frac{1}{r}} \tilde{S} = \frac{1}{r} \max(0, Su - K) + \frac{1}{r} \max(0, Sd - K).$$

Here, $1/r$ is the discount factor.

Note that the Black-Scholes formula is of the same form but with specific formulas for $\alpha$ and $\beta$:

$$\alpha = \frac{r - d}{u - d}$$
$$\beta = \frac{u - r}{u - d}$$

Many consider it a feature that the Black-Scholes formula does not depend on prespecified probabilities. In my opinion it is proof of a bug in the model. Which formula do you believe?