Quadratic Programming

The Markowitz Model for Portfolio Optimization
What is Quadratic Programming?

Objective is convex quadratic.

Constraints must be linear.

\[
\begin{align*}
\text{minimize} & \quad c^T x + \frac{1}{2} x^T Q x \\
\text{subject to} & \quad A x \geq b \\
& \quad x \geq 0
\end{align*}
\]

Without loss of generality, the matrix $Q$ of quadratic coefficients may be assumed symmetric.

The matrix $Q$ must be positive semidefinite. Any of the following conditions characterize positive semidefiniteness:

- $\xi^T Q \xi \geq 0$ for all $\xi$.
- All eigenvalues of $Q$ are nonnegative.
- There exists a matrix $F$ such that $Q = F^T F$. 
Markowitz Shares the 1990 Nobel Prize

**Press Release - The Sveriges Riksbank (Bank of Sweden) Prize in Economic Sciences in Memory of Alfred Nobel**

KUNGL. VETENSKAPS AKADEMIEN
THE ROYAL SWEDISH ACADEMY OF SCIENCES

16 October 1990

**THIS YEAR’S LAUREATES ARE PIONEERS IN THE THEORY OF FINANCIAL ECONOMICS AND CORPORATE FINANCE**

The Royal Swedish Academy of Sciences has decided to award the 1990 Alfred Nobel Memorial Prize in Economic Sciences with one third each, to

Professor **Harry Markowitz**, City University of New York, USA,
Professor **Merton Miller**, University of Chicago, USA,
Professor **William Sharpe**, Stanford University, USA,

for their pioneering work in the theory of financial economics.

**Harry Markowitz** is awarded the Prize for having developed the theory of portfolio choice;
**William Sharpe**, for his contributions to the theory of price formation for financial assets, the so-called, *Capital Asset Pricing Model* (CAPM); and
**Merton Miller**, for his fundamental contributions to the theory of corporate finance.

**Summary**

Financial markets serve a key purpose in a modern market economy by allocating productive resources among various areas of production. It is to a large extent through financial markets that saving in different sectors of the economy is transferred to firms for investments in buildings and machines. Financial markets also reflect firms’ expected prospects and risks, which implies that risks can be spread and that savers and investors can acquire valuable information for their investment decisions.

The first pioneering contribution in the field of financial economics was made in the 1950s by Harry Markowitz who developed a theory for households’ and firms’ allocation of financial assets under uncertainty, the so-called theory of portfolio choice. This theory analyzes how wealth can be optimally invested in assets which differ in regard to their expected return and risk, and thereby also how risks can be reduced.

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Let’s describe the Markowitz model...
## Historical Data

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**Notation:** $R_j(t) = \text{return on investment } j \text{ in time period } t.$
**Risk vs. Reward**

**Reward**—estimated using historical means:

\[
\text{reward}_j = \frac{1}{T} \sum_{t=1}^{T} R_j(t).
\]

**Risk**—estimated using historical variances:

\[
\text{risk}_j = \frac{1}{T} \sum_{t=1}^{T} (R_j(t) - \text{reward}_j)^2.
\]

**Hedging**

**Investment A:** up 20%, down 10%, equally likely.

**Investment B:** up 20%, down 10%, equally likely.

**Correlation:** up years for A are down years for B and vice versa.

**Portfolio—half in A, half in B:** up 5% every year!
**Portfolios**

**Fractions:** \( x_j \) = fraction of portfolio to invest in \( j \).

**Portfolio’s Historical Returns:**

\[
R(t) = \sum_j x_j R_j(t)
\]

**Portfolio’s Reward:**

\[
\text{reward}(x) = \frac{1}{T} \sum_{t=1}^{T} R(t) = \frac{1}{T} \sum_{t=1}^{T} \sum_j x_j R_j(t)
\]

**Portfolio’s Risk:**

\[
\text{risk}(x) = \frac{1}{T} \sum_{t=1}^{T} (R(t) - \text{reward}(x))^2
\]

\[
= \frac{1}{T} \sum_{t=1}^{T} \left( \sum_j x_j R_j(t) - \frac{1}{T} \sum_{s=1}^{T} \sum_j x_j R_j(s) \right)^2
\]

\[
= \frac{1}{T} \sum_{t=1}^{T} \left( \sum_j x_j \left( R_j(t) - \frac{1}{T} \sum_{s=1}^{T} R_j(s) \right) \right)^2
\]

\[
= \frac{1}{T} \sum_{t=1}^{T} \left( \sum_j x_j (R_j(t) - \text{reward}_j) \right)^2
\]

**Note:** \( \text{risk}(x) \) is quadratic in the \( x_j \)’s.
The Markowitz Model

Decision Variables: the fractions $x_j$.

Objective: maximize return, minimize risk.

Fundamental Lesson: can’t simultaneously optimize two objectives.

Compromise: maximize a combination of reward and risk:

$$\text{reward}(x) - \mu \text{risk}(x)$$

Parameter $\mu$ is called risk aversion parameter.

$$0 \leq \mu < \infty$$

Large value for $\mu$ puts emphasis on risk minimization.

Small value for $\mu$ puts emphasis on reward maximization.

Constraints:

$$\sum_{j} x_j = 1$$

$$x_j \geq 0 \quad \text{for all investments } j$$

Quadratic Program:

Constraints are linear.

Objective is quadratic.
**Efficient Frontier**

Varying $\mu$ produces the so-called efficient frontier.

Portfolios on the efficient frontier are reasonable.

Portfolios not on the efficient frontier can be strictly improved.

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AMPL Model

set A;  # asset categories
set T := {1973..1994};  # years

param lambda default 200;

param R {T,A};

param mean {j in A} := ( sum{i in T} R[i,j] )/card(T);

param Rtilde {i in T, j in A} := R[i,j] - mean[j];

param Cov {j in A, k in A} := sum {i in T} (Rtilde[i,j]*Rtilde[i,k]) / card(T);

var x{A} >=0;

minimize lin_comb:
    lambda * sum{i in T} (sum{j in A} Rtilde[i,j]*x[j])^2 / card{T} - sum{j in A} mean[j]*x[j];

subject to tot_mass:
    sum{j in A} x[j] = 1;

data;

set A :=
    US_3-MONTH_T-BILLS US_GOVN_LONG_BONDS SP_500 WILSHIRE_5000 NASDAQ_COMPOSITE LEHMAN_BROTHERS_CORPORATE_BONDS_INDEX EAFE GOLD;

param R:
    US_3-MONTH_T-BILLS US_GOVN_LONG_BONDS SP_500 WILSHIRE_5000 NASDAQ_COMPOSITE LEHMAN_BROTHERS_CORPORATE_BONDS_INDEX EAFE GOLD :=
1973  1.075  0.942  0.852  0.815  0.698  1.023  0.851  1.677
1974  1.084  1.020  0.735  0.716  0.662  1.002  0.768  1.722
1975  1.061  1.056  1.371  1.385  1.318  1.123  1.354  0.760
    .
    .
1994  1.045  0.889  1.012  0.999  0.968  0.965  1.078  0.990
;
solve;

printf: "--------------------------------------------------\n";
printf: " Asset    Mean    Variance \n";
printf {j in A}: "%45s %10.7f %10.7f \n",
    j, mean[j], sum{i in T} Rtilde[i,j]^2 / card(T);
printf: "\n";
printf: "Optimal Portfolio: Asset Fraction \n";
printf {j in A: x[j] > 0.001}: "%45s %10.7f \n", j, x[j];

printf: "Mean = %10.7f, Variance = %10.5f \n",
   sum{j in A} mean[j]*x[j],
   sum{i in T} (sum{j in A} Rtilde[i,j]*x[j])^2 / card(T);
Interior-Point Methods for Quadratic Programming

Start with an optimization problem—in this case QP:

\[
\begin{align*}
\text{minimize} \quad & c^T x + \frac{1}{2}x^T Q x \\
\text{subject to} \quad & Ax \geq b \\
& x \geq 0
\end{align*}
\]

Use slack variables to make all inequality constraints into nonnegativities:

\[
\begin{align*}
\text{minimize} \quad & c^T x + \frac{1}{2}x^T Q x \\
\text{subject to} \quad & Ax - w = b \\
& x, w \geq 0
\end{align*}
\]

Replace nonnegativity constraints with logarithmic barrier terms in the objective:

\[
\begin{align*}
\text{minimize} \quad & c^T x + \frac{1}{2}x^T Q x - \mu \sum_j \log x_j - \mu \sum_i \log w_i \\
\text{subject to} \quad & Ax - w = b
\end{align*}
\]

Introduce Lagrange multipliers to form Lagrangian:

\[
c^T x + \frac{1}{2}x^T Q x - \mu \sum_j \log x_j - \mu \sum_i \log w_i - y^T (Ax - w - b)
\]

Set derivatives to zero:

\[
\begin{align*}
c + Q x - \mu X^{-1} e - A^T y &= 0 \\
-\mu W^{-1} e + y &= 0 \\
-Ax + w + b &= 0
\end{align*}
\]
Introduce dual complementary variables:

\[ z = \mu X^{-1}e \]

Rewrite system:

\[
\begin{align*}
c + Qx - z - A^T y &= 0 \\
XZe &= \mu e \\
WYe &= \mu e \\
b - Ax + w &= 0
\end{align*}
\]

Introduce step directions: \( \Delta x, \Delta y, \Delta w, \Delta z \).

Write the above equations for \( x + \Delta x, y + \Delta y, w + \Delta w \), and \( z + \Delta z \):

\[
\begin{align*}
c + Q(x + \Delta x) - (z + \Delta z) - A^T (y + \Delta y) &= 0 \\
(X + \Delta X)(Z + \Delta Z)e &= \mu e \\
(W + \Delta W)(Y + \Delta Y)e &= \mu e \\
b - A(x + \Delta x) + (w + \Delta w) &= 0
\end{align*}
\]

Rearrange with “delta” variables on left and drop nonlinear terms on left:

\[
\begin{align*}
Q\Delta x - \Delta z - A^T \Delta y &= -c - Qx + z + A^T y \\
Z\Delta x + X\Delta z &= \mu e - ZXe \\
W\Delta y + Y\Delta w &= \mu e - WYe \\
-A\Delta x + \Delta w &= -b + Ax - w
\end{align*}
\]

This is a linear system of \( 2m + 2n \) equations in \( 2m + 2n \) unknowns.

Solve’em.
Yadda, yadda, yadda.
Parting Comments

The matrix $Q$ must be positive semidefinite (psd), which means that all of the eigenvalues of $Q$ must be nonnegative.

Such QPs are called convex quadratic programming problems.

Nonconvex QPs are as hard to solve as integer programs. Many local optima, which is best? Hmm.

The Markowitz model is a convex QP (the covariance matrix is always psd).