1. **Piecewise Linear Approximation.** Given real numbers $b_1 < b_2 < \cdots < b_k$, let $f$ be a continuous function on $\mathbb{R}$ that is linear on each interval $[b_i, b_{i+1}]$, $i = 0, 1, \ldots, k$ (for convenience we let $b_0 = -\infty$ and $b_{k+1} = \infty$). Such a function is called *piecewise linear* and the numbers $b_i$ are called *breakpoints*. Piecewise linear functions are often used to approximate (continuous) nonlinear functions. The purpose of this exercise is to show how and why.

(a) Every piecewise linear function can be written as a sum of a constant plus a linear term plus a sum of absolute value terms:

$$f(x) = d + a_0 + \sum_{i=1}^{k} a_i |x - b_i|.$$  

Let $c_i$ denote the slope of $f$ on the interval $[b_i, b_{i+1}]$. Derive an explicit expression for each of the $a_j$’s (including $a_0$) in terms of the $c_i$’s.

(b) In terms of the $c_i$’s, give necessary and sufficient conditions for $f$ to be convex.

(c) In terms of the $a_j$’s, give necessary and sufficient conditions for $f$ to be convex.

(d) Assuming that $f$ is convex and is a term in the objective function for a linearly constrained optimization problem, derive an equivalent linear programming formulation involving at most $k$ extra variables and constraints.

(e) Repeat the first four parts of this problem using $\max(x - b_i, 0)$ in place of $|x - b_i|$.

2. Let $f$ be the function of 2 real variables defined by

$$f(x, y) = x^2 - 2xy + y^2.$$  

Show that $f$ is convex.

3. A function $f$ of 2 real variables is called a *monomial* if it has the form

$$f(x, y) = x^m y^n$$  

for some nonnegative integers $m$ and $n$. Which monomials are convex?

4. Let $\phi$ be a convex function of a single real variable. Let $f$ be a function defined on $\mathbb{R}^n$ by the formula

$$f(x) = \phi(a^T x + b),$$  

where $a$ is an $n$-vector and $b$ is a scalar. Show that $f$ is convex.
5. Which of the following functions are convex (assume that the domain of the function is all of \( \mathbb{R}^n \) unless specified otherwise)?
   (a) \( 4x^2 - 12xy + 9y^2 \)
   (b) \( x^2 + 2xy + y^2 \)
   (c) \( x^2y^2 \)
   (d) \( x^2 - y^2 \)
   (e) \( e^{x-y} \)
   (f) \( e^{x^2-y^2} \)
   (g) \( \frac{x^2}{y} \) on \( \{ (x, y) : y > 0 \} \)

6. Given a symmetric square matrix \( A \), the quadratic form \( x^T A x = \sum_{i,j} a_{ij} x_i x_j \) generalizes the notion of the square of a variable. The generalization of the notion of the fourth power of a variable is an expression of the form

\[
f(x) = \sum_{i,j,k,l} a_{ijkl} x_i x_j x_k x_l.
\]

The four-dimensional array of numbers \( A = \{ a_{ijkl} : 1 \leq i \leq n, 1 \leq j \leq n, 1 \leq k \leq n, 1 \leq l \leq n \} \) is called a 4-tensor. As with quadratic expressions, we may assume that \( A \) is symmetric:

\[
a_{ijkl} = a_{jikl} = \cdots = a_{lkji}
\]

(i.e., given \( i, j, k, l \), all \( 4! = 24 \) permutations must give the same value for the tensor).

(a) Give conditions on the 4-tensor \( A \) to guarantee that \( f \) is convex.

(b) Suppose that some variables, say \( y_i \)'s, are related to some other variables, say \( x_j \)'s, in a linear fashion:

\[
y_i = \sum_j f_{ij} x_j.
\]

Express \( \sum_i y_i^4 \) in terms of the \( x_j \)'s. In particular, give an explicit expression for the 4-tensor and show that it satisfies the conditions derived in part (a).
7. Consider the problem

\[
\begin{align*}
\text{minimize} & \quad ax_1 + x_2 \\
\text{subject to} & \quad \sqrt{e^2 + x_1^2} \leq x_2.
\end{align*}
\]

where \(-1 < a < 1\).

(a) Graph the feasible set: \(\{(x_1, x_2) : \sqrt{e^2 + x_1^2} \leq x_2\}\). Is the problem convex?

(b) Following the steps in the middle of p. 391 of the text, write down the first-order optimality conditions for the barrier problem associated with barrier parameter \(\mu > 0\).

(c) Solve explicitly the first-order optimality conditions. Let \((x_1(\mu), x_2(\mu))\) denote the solution.

(d) Graph the central path, \((x_1(\mu), x_2(\mu))\), as \(\mu\) varies from 0 to \(\infty\).