1. Consider the saddle point problem: find $u \in V$ and $p \in W$ such that
\begin{align}
  a(u, v) + b(v, p) &= g(v), \quad v \in V, \quad (1) \\
  b(u, w) &= f(w), \quad w \in W, \quad (2)
\end{align}
and its approximation: find $u \in V \subset V$ and $p \in W \subset W$ such that
\begin{align}
  a(u_h, v_h) + b(v_h, p_h) &= g(v_h), \quad v_h \in V_h, \quad (3) \\
  b(u_h, w_h) &= f(w_h), \quad w_h \in W_h. \quad (4)
\end{align}
Let
\begin{align*}
  V^0 &= \{ v \in V : b(v, w) = 0 \quad \forall \ w \in W \}, \\
  V_h^0 &= \{ v_h \in V_h : b(v_h, w) = 0 \quad \forall \ w_h \in W_h \}.
\end{align*}
Assume that all assumptions of Theorem 3 from the saddle point theory hold. Assume also that $V_h^0 \subset V^0$. Prove that
\[
  \|u - u_h\|_V \leq C \inf_{v_h \in V_h} \|u - v_h\|_V.
\]

2. Consider the MFEM discretization of the elliptic problem
\[
  -\nabla \cdot K \nabla p = f \quad \text{in } \Omega, \quad p = 0 \quad \text{on } \partial \Omega,
\]
based on simplicial RT$_k$ elements.

a) Show that, if for any $w_h \in W_h$ there exists $v_h \in V_h$ such that $\nabla \cdot v_h = w_h$ and $\|v_h\|_V \leq C\|w_h\|_W$, then the discrete inf-sup condition holds with a constant independent of $h$.

b) For any $w_h \in W_h$, construct $v_h \in V_h$ having the above properties.

3. Consider the boundary value problem
\[
  -\nabla \cdot K \nabla p = f \quad \text{in } \Omega, \quad -K \nabla p \cdot n = 0 \quad \text{on } \partial \Omega, \quad (5)
\]
where $K$ is symmetric and positive definite tensor with $L^\infty(\Omega)$ components and $f \in L^2(\Omega)$ satisfies $\int_\Omega f = 0$.

a) Derive the dual mixed variational formulation of (5) and show that it has a unique solution. Hint: take $W = L^2_0$. 

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b) Derive the mixed finite element approximation to a) based on the Raviart-Thomas spaces RTₖ on affine elements and show that it has a unique solution.

c) Show that
\[ \| \mathbf{u} - \mathbf{u}_h \| \leq C \| \mathbf{u} - \Pi_h \mathbf{u} \| \]

and
\[ \| \nabla \cdot (\mathbf{u} - \mathbf{u}_h) \| \leq \| \nabla \cdot (\mathbf{u} - \Pi_h \mathbf{u}) \|. \]

d) Show that
\[ \| p - p_h \| \leq C(\| p - Q_h p \| + \| \mathbf{u} - \mathbf{u}_h \|) \]

e) (Bonus) Show that
\[ \| Q_h p - p_h \| \leq C h^{k+2} \]

Hint: Consider the auxiliary problem
\[ -\nabla \cdot K \nabla \varphi = Q_h p - p_h \quad \text{in } \Omega, \quad -K \nabla \varphi \cdot n = 0 \quad \text{on } \partial \Omega. \]

4. Consider the matrix
\[ S = \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \]
arising in the MFEM discretization of the elliptic problem
\[ -\nabla \cdot K \nabla p = f \quad \text{in } \Omega, \quad p = 0 \quad \text{on } \partial \Omega, \]

where \( K \) is a symmetric and positive definite tensor.

a) Prove that \( A \) is symmetric and positive definite.

b) Prove that the Schur complement \( R = BA^{-1}B^T \) is symmetric and positive definite.