Consider the boundary value problem from Homework 5

\[-\nabla \cdot A \nabla u = f \quad \text{in } \Omega \quad (1)\]

\[u = g_D \quad \text{on } \Gamma_D \quad (2)\]

\[\gamma u + A \nabla u \cdot n = g_R \quad \text{on } \Gamma_R, \quad (3)\]

where $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ is a convex polygon, $\partial \Omega = \Gamma_D \cup \Gamma_R$, $|\Gamma_D| > 0$, $0 \leq \gamma(x) \leq \gamma_1$, and $A(x)$ is a $d \times d$ symmetric matrix satisfying

\[\alpha_0 \xi^T \xi \leq \xi^T A(x) \xi \leq \alpha_1 \xi^T \xi \quad \forall \xi \in \mathbb{R}^d, \quad \forall x \in \Omega,\]

where $0 < \alpha_0 \leq \alpha_1 < \infty$.

a) Formulate the Galerkin finite element method for the approximation of (1)–(3) based on continuous piecewise polynomials of degree $k$ on simplicial grids.

b) Prove that the Galerkin finite element method has a unique solution. (You may refer to results proved in Homework 5.)

c) Derive an optimal order error estimate in the $H^1(\Omega)$-norm. (You may use the approximation result for piecewise polynomials stated in class.)

d) Use the “Nitsche lift” to prove an optimal error estimate in the $L^2(\Omega)$-norm. State carefully any assumptions needed for the argument.