Math 2602 Homework 1

Due February 1, 2008

1. Consider the boundary value problem

$$-\nabla \cdot K\nabla p = f \quad \text{in } \Omega, \qquad p = g_D \quad \text{on } \partial\Omega,$$

where K is symmetric and positive definite tensor with $L^\infty(\Omega)$ components, $f \in L^2(\Omega)$, and $g_D \in H^{1/2}(\partial\Omega)$. Derive the dual mixed variational formulation and the corresponding inf-sup (saddle point) problem. Show that the two formulations are equivalent.

2. Find the first and the second distributional derivatives of $f(x) = |x|$. **Hint:** Divide the real axis into two parts at $x = 0$.

3. Let $\Omega = \cup_{i=1}^n \Omega_i$, where Ω_i are mutually disjoint. Suppose that

$$\mathbf{u}_i := \mathbf{u}|_{\Omega_i} \in H(\text{div}; \Omega_i), \quad i = 1, \dots n$$

and

$$\mathbf{u}_i \cdot \mathbf{n}_i + \mathbf{u}_j \cdot \mathbf{n}_j = 0 \text{ on } \partial\Omega_i \cap \partial\Omega_j \quad i, j = 1 \dots n,$$

where $\mathbf{n}_i$ is the outward unit normal to $\partial\Omega_i$. Prove that

$$\mathbf{u} \in H(\text{div}; \Omega).$$

4. Let $f \in H^k(\Omega)$, $k \in \boldsymbol{Z}$. Show that

$$\frac{\partial f}{\partial x_i} \in H^{k-1}(\Omega).$$

5. Let $f \in H^{-k}(\Omega)$, $k \in \boldsymbol{N}$. Show that there exist functions $g_\alpha \in L^2(\Omega)$ such that

$$f = \sum_{|\alpha| \le k} D^\alpha g_\alpha.$$