Math 3072 Finite Element Methods

Homework 4, Due March 14, 2007

1. Consider the boundary value problem

\[-\nabla \cdot A \nabla u + b \cdot \nabla u + a_0 u = f \text{ in } \Omega \]
\[u = 0 \text{ on } \Gamma_D \]
\[A \nabla u \cdot n = g_N \text{ on } \Gamma_N,\]
where \(\partial \Omega = \Gamma_D \cup \Gamma_N, \|\Gamma_D\| > 0,\) and \(A(x)\) is a \(d \times d\) symmetric matrix satisfying

\[\alpha_0 \xi^T \xi \leq \xi^T A(x) \xi \leq \alpha_1 \xi^T \xi \quad \forall \xi \in \mathbb{R}^d, \quad \forall x \in \Omega,\]
where \(0 < \alpha_0 \leq \alpha_1 < \infty.\) Assume also that

\[a_0 - \frac{1}{2} \nabla \cdot b \geq 0 \quad \text{and} \quad b \cdot n \geq 0 \text{ on } \Gamma_N.\]

a) Derive the weak formulation of (1)–(3).

b) Prove that there exists a unique weak solution.

c) Formulate the Galerkin finite element method for approximating the weak solution based on continuous piecewise linear polynomials on a triangular mesh. Prove that it has a unique solution. Discuss the properties of the resulting linear system.

d) Assume that

\[\inf_{v \in S_h} \|u - v\|_{H^1} \leq Ch|u|_{H^2}.\]

Prove that

\[\|u - u_h\|_{H^1} \leq Ch|u|_{H^2}.\]

Note: You are not allowed to cite the Cea’s Theorem.

e) Use the Nitsche Lift to prove that

\[\|u - u_h\|_{L^2} \leq Ch^2|u|_{H^2}.\]

Note: State the elliptic regularity assumption needed for this argument.
Note: Be careful in the derivation of the dual bilinear form
\[ a^*(u, v) = a(v, u). \]

2. Consider the Laplace’s operator \(-\Delta u\) in \(R^2\).
   a) Calculate the element stiffness matrix for linear polynomials on the reference triangle.
   b) For the attached mesh, calculate all non-zero elements of the 5th row of the stiffness matrix.