MATH 0250 MIDTERM II (sample) Mar 30, 2001

System of First Order Differential Equations with Constant Coefficients

1. *(Eigenproblem)* Find an eigenpair for the matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

2. *(Fundamental Solution Matrix)* The matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has an eigenpair $\left\{ \mathbf{i}, \begin{bmatrix} 1 \\ \mathbf{i} \end{bmatrix} \right\}$. Find the fundamental solution matrix $\Psi(t)$ to the differential system $\mathbf{x}' = \mathbf{A}\,\mathbf{x}$ with the property that $\Psi(0)$ is the identity matrix.

3. *(Undetermined Coefficients)* Find a particular solution to $\mathbf{x}' = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} \mathbf{x} + e^{-t}\cos(2t) \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

4. *(Variation of Parameters)* Let $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$ and $\Psi(t) = e^t \begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix}$. (a) Show that $\Psi(t)$ is a fundamental solution matrix to $\mathbf{y}' = \mathbf{A}\,\mathbf{y}$. (b) Find the solution to the initial value problem $\mathbf{x}' = \mathbf{A}\,\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

5. *(Initial Value Problem)* Solve $\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}$, $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Second Order Linear Differential Equation with Constant Coefficients

1. *(Homogeneous equations)* Find the general solution to
(a) $u'' - 3u' + 2u = 0$, (b) $u'' + u' + 2u = 0$, (c) $u'' - 2u' + u = 0$

2. *(Undetermined Coefficients)* Find a particular solution to
(a) $u'' + 2u = 1 + 2t + t^2$ (b) $u'' - 2u' + u = e^t$ (c) $u'' + u = 4\sin t$.

3. *(Variation of Parameters)* Find general solution to $u'' + u = \sec t$

4. *(Initial Value Problem)* Solve $u'' + u = 2$, $u(0) = 0$, $u'(0) = 1$.

Some Theoretical Proofs

1. Suppose that $\mathbf{A}$ is a real matrix and $\mathbf{y}(t)$ is a complex vector valued function solving $\mathbf{y}' = \mathbf{A}\mathbf{y}$. Show that both the real and imaginary parts of $\mathbf{y}$ solve $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

2. Assume that $\mathbf{A}$ is a constant matrix and $\Phi(t)$ is a solution matrix to $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Let $\Psi(t) = \Phi'(t)$. Show that $\Psi(t)$ is also a solution matrix to $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

3. Let $\mathcal{L}$ be a linear differential operator. Assume that $\mathcal{L}(u_1) = 0$, $\mathcal{L}(u_2) = 0$ and $\mathcal{L}(u_p) = f$. Show that for any constants c_1 and c_2, the function $u = c_1 u_1 + c_2 u_2 + u_p$ solves $\mathcal{L}(u) = f$.

4. Let $\mathcal{L}(u) = u'' + pu' + qu$ where p and q are real functions. Assume that $w = (t - \mathbf{i}t^2)e^{\mathbf{i}t}$ solves $\mathcal{L}(w) = te^{\mathbf{i}t}$. Find a particular solution to $\mathcal{L}(u) = t\cos t + 2t\sin t$.