

MATH 0230 MIDTERM II ANSWER

1. Find the integrals for the following.

(1) The arclength of $x = y^2$, $0 \leq y \leq 3$.

$$L = \int_0^3 \sqrt{1 + (2y)^2} dy.$$

(2) The work to pull a spring of constant 2000 (N/m) from its equilibrium position by 5 (m).

$$W = \int_0^5 2000x dx.$$

(3) The y -coordinate of the centroid of a lamina occupying the region bounded by $y = 0$ and $y = 2x - x^2$.

$$\bar{y} = \frac{\frac{1}{2} \int_0^2 \{[2x - x^2]^2 - 0^2\} dx}{\int_0^2 \{[2x - x^2] - 0\} dx}$$

2. (a) Let $f(x) = (1 + x)^{1/3}$. Find a polynomial $P(x)$ of degree as small as possible such that $|f(x) - P(x)| \leq 10^{-5}$ for all x in $[0, 0.1]$.

Solution. When $0 \leq x \leq 0.1$,

$$|f(x) - T_n(x, 0)| \leq \frac{M|x|^{n+1}}{(n+1)!} \leq \frac{M10^{-(n+1)}}{(n+1)!}.$$

$$f'(x) = \frac{1}{3}(1+x)^{-2/3}, \quad f'(0) = \frac{1}{3}$$

$$f''(x) = -\frac{2}{9}(1+x)^{-5/3}, \quad f''(0) = -\frac{2}{9}$$

$$f'''(x) = \frac{10}{27}(1+x)^{-8/3}, \quad f'''(0) = \frac{10}{27},$$

$$f''''(x) = -\frac{80}{81}(1+x)^{-8/3}, \quad M = \frac{80}{81}$$

$$|f(x) - T_3(x, 0)| \leq \frac{80}{81} \frac{10^{-4}}{4!} < 5 \times 10^{-6}$$

$$P(x) = T_3(x; 0) = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3.$$

When $x = 0.1$, the last term $\frac{5}{81}10^{-3} > 6 \times 10^{-5}$, so $T_2(x; 0)$ is not accurate enough.

3. The position vector of a moving particle at time t is $\mathbf{r}(t) = \langle \cos t, t \rangle$. At $t = 0$, find the velocity, speed, acceleration, unit tangent vector, and tangential acceleration.

$\mathbf{r}'(t) = \langle -\sin t, 1 \rangle$, $\mathbf{r}''(t) = \langle -\cos t, 0 \rangle$. At $t = 0$,
velocity $\mathbf{v} = \mathbf{r}'(0) = \langle 0, 1 \rangle$;

speed $v = \|\mathbf{r}'(0)\| = 1$;

acceleration $\mathbf{a} = \mathbf{r}''(0) = \langle -1, 0 \rangle$;

unit tangent $\mathbf{T} = \mathbf{v}/v = \langle 0, 1 \rangle$;

scalar tangential acceleration $a_T = \mathbf{a} \cdot \mathbf{T} = 0$;

tangential acceleration $\mathbf{a}_T = a_T \mathbf{T} = \langle 0, 0 \rangle$;

normal acceleration $\mathbf{a}_N = \mathbf{a} - a_T \mathbf{T} = \langle -1, 0 \rangle$;

scalar normal acceleration $a_N = \|\mathbf{a}_N\| = 1$;

unit normal $\mathbf{N} = \mathbf{a}_N/a_N = \langle -1, 0 \rangle$;

curvature $\kappa = a_N/v^2 = 1$.

4. Let $\{a_n\}_{n=0}^\infty$ be defined by $a_0 = 1, a_n = a_{n-1}/2$ for all $n \geq 1$. Also let $s_n = \sum_{k=0}^n a_k$. Find (i) a_3 , (ii) s_3 , and (iii) $\lim_{n \rightarrow \infty} s_n$.

(i) $a_0 = 1, a_1 = \frac{1}{2}, a_2 = \frac{1}{4}, a_3 = \frac{1}{8}$.

(ii) $s_3 = a_0 + a_1 + a_2 + a_3 = \frac{15}{8}$.

(iii) $\lim_{n \rightarrow \infty} s_n = 1 + \frac{1}{2} + \frac{1}{4} + \dots = 2$.

5. Find a_0, \dots, a_8 in Taylor expansion $\int_0^x \sin(z^2) dz$.

$$\begin{aligned} \int_0^x \sin(z^2) dz &= \int_0^x \left\{ z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} + \dots \right\} dz \\ &= 0 + 0x + 0x^2 + \frac{1}{3}x^3 + 0x^4 + 0x^5 + 0x^6 \\ &\quad - \frac{1}{42}x^7 + 0x^8 + \dots \end{aligned}$$

$$a_3 = \frac{1}{3}, \quad a_7 = -\frac{1}{42},$$

$$0 = a_0 = a_1 = a_2 = a_4 = a_5 = a_6 = a_8,$$

6. Find all those real value x such that the series $\sum_{n=1}^\infty \frac{(2x)^n}{\sqrt{n}}$ is convergent.

Solution. To use ratio test, we calculate

$$\lim_{n \rightarrow \infty} \frac{|(2x)^{n+1}/\sqrt{n+1}|}{|(2x)^n/\sqrt{n}|} = |2x|.$$

When $|2x| < 1$, series converges.

When $|2x| > 1$, series diverges.

When $2x = 1$, the series, $\sum_{n=1}^\infty \frac{1}{\sqrt{n}}$, diverges.

When $2x = -1$, the series, $\sum_{n=1}^\infty \frac{(-1)^n}{\sqrt{n}}$, is convergent, by alternating series test.

\therefore The series converges when $-\frac{1}{2} \leq x < \frac{1}{2}$.

7. Test for convergence:

$$\sum_{n=1}^\infty \frac{\ln n}{n}, \quad \sum_{n=1}^\infty \frac{(-1)^n n}{100n + 1000}, \quad \sum_{n=100}^\infty \frac{1000n}{n^3 - 6n - 10}$$

(i) $\because \frac{\ln n}{n} \geq \frac{1}{n}$ for $n \geq 3$ and $\sum_{n=1}^\infty \frac{1}{n} = \infty$,
 \therefore the series diverges, by comparison test.

(ii) $\because \lim_{n \rightarrow \infty} \left| \frac{(-1)^n n}{100n + 1000} \right| = \frac{1}{100} \neq 0$,
 \therefore the series is divergent.

(iii) Use limit comparison test to compare it with the convergent series $\sum \frac{1}{n^2}$.

$$\lim_{n \rightarrow \infty} \frac{1000n}{n^3 - 6n - 10} * n^2 = 1000.$$

The series is convergent.