Lecture Notes On

PARTIAL DIFFERENTIAL EQUATIONS

PART II

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To my friends

PREFACE

This notes is based on the course Math 2901, *Partial Differential Equations Part II*, offered in the Spring of 2001 at the University of Pittsburgh.

The first LaTeX draft of this notes was prepared by the graduate students. In particular, Mr. Edward Krisner, Mr. Atife Caglar, Mr Jonathan Holland, Ms. Argus Dunca and Mr Jason Morris carefully edited their class notes for Chapters 1 to 5 respectively. Here I thank them for their effort, their patience, and especially their hunger for the PDE knowledge which encouraged me greatly.

I take the full responsibility on the mistakes in the notes.

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Contents

Chapter 1

SOBOLEV SPACE

In studying differential equations, a differentiation can be regarded as an operator from one function space to another. Solving a (linear) differential equation is equivalent to find the inverse of an operator. In general, the natural function spaces involved are the Sobolev spaces. Indeed, the theory of Sobolev space has become inseparable with the study of partial differential equations.

In this chapter, we shall introduce the basic theory on Sobolev spaces. For simplicity, we may choose not to present certain theorems in their strongest forms.

1.1 Preliminary

For convenience of the readers, we review a few basic mathematical concepts.

1. *Vector Space*

2. *Measure*

3. *Metric Space*

4. *Banach Space*

5. *Completion*

6. *Hilbert Space*

   Clearly, a Hilbert space is more preferable than a Banach space.

7. *Functional:* A (linear) map from a banach space to $\mathbb{R}$ or $\mathbb{C}$.

8. *Dual space:* The space consists of all linear functionals.

   For a Hilbert space $\mathbf{H}$, its dual $\mathbf{H}^*$ is isomorphic to $\mathbf{H}$ (the Riesz representation theorem). In applications to PDEs, it is definitely needed to distinguish H and H^*.

9. *Operator:* A map from one space of functions to another.

10. *Adjoint of an operator*

11. *Topological Space*

12. *Compact set*

13. *Precompact set*

14. *Compact Operator*

Let $\mathbf{X}$ and $\mathbf{Y}$ be any two Banach space and $\Omega \subset \mathbf{X}$ be a subset. A function $u : \Omega \to \mathbf{Y}$ is called *continuous* if the pre–image in Ω of every open set is relatively open. We use $C(\Omega; \mathbf{Y})$ to denote all continuous functions from Ω to $\mathbf{Y}$. When $\mathbf{Y}$ is clear, we use $C(\Omega)$ to denote $C(\Omega; Y)$.

In this chapter, $n \geq 1$ denote the space dimension, k a non–negative integer, $\gamma \in (0, 1]$ a real number, and $\alpha = (\alpha_1, \cdots, \alpha_n)$, where all α_i are non–negative integers, a multi-index, whose order is denoted by $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

1.2 The Hölder Spaces

1.2.1 Definition

Definition 1.1. *Let Ω be an open set in $\mathbb{R}^n$ and $u \in C(\bar{\Omega}; \mathbb{R}^m)$. We write*

$$\|u\|_{C(\bar{\Omega})} = \sup_{x \in \Omega} |u(x)|, \qquad \|u\|_{C^k(\bar{\Omega})} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{C(\bar{\Omega})},$$

$$[u]_{\gamma,\bar{\Omega}} = \sup_{x,y \in \Omega, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\gamma}, \qquad \|u\|_{C^{k+\gamma}(\bar{\Omega})} = \|u\|_{C^k(\bar{\Omega})} + \sum_{|\alpha| = k} [D^\alpha u]_{\gamma,\bar{\Omega}}.$$

Definition 1.2. *The Hölder space $C^{k+\gamma}(\bar{\Omega})$ consists of all functions $u \in C^k(\bar{\Omega})$ with finite $\|u\|_{C^{k+\gamma}(\bar{\Omega})}$ norm.*

For simplicity, we often write $\| \cdot \|_{C^{k+\gamma}(\bar{\Omega})}$ as $\| \cdot \|_{k+\gamma,\bar{\Omega}}$. Also, when $\gamma = 1$, we write $C^{k+\gamma}$ as $C^{k,1}$, since C^{k+1} and $C^{k,1}$ are different function spaces. For convenience, $C^{k+\gamma} = C^k$ when $\gamma = 0$.

Theorem 1.1. *The space $C^{k+\gamma}(\bar{\Omega})$ is a Banach space.*

Theorem 1.2 (Approximation). *If $\gamma \in [0, 1)$, then $C^{k+\gamma}(\overline{\Omega})$ is the completion of $C^\infty(\bar{\Omega})$ under the norm $\| \cdot \|_{C^{k+\gamma}(\bar{\Omega})}$.*

Theorem 1.3 (Interpolation). *Let $\gamma_1 < \beta < \gamma_2$. Then*

$$\| \cdot \|_{\beta,\bar{\Omega}} \leq \| \cdot \|_{\gamma_2,\bar{\Omega}}^{(\beta-\gamma_1)/(\gamma_2-\gamma_1)} \, \| \cdot \|_{\gamma_1,\bar{\Omega}}^{(\gamma_2-\beta)/(\gamma_2-\gamma_1)}$$

Homework 1.1. *Assume $f \in C^\alpha(\mathbb{R}^m; \mathbb{R}^n)$ and $g \in C^\beta(\Omega; \mathbb{R}^m)$ where $0 \leq \alpha, \beta \leq 1$. Show that $f(g(\cdot)) \in C^{\alpha\beta}(\Omega; \mathbb{R}^n)$.*

Homework 1.2. *Suppose that* $f, g \in C^{k+\gamma}$. *Show that* $fg \in C^{k+\gamma}$ *and* $\|fg\|_{C^{k+\gamma}} \leq \|f\|_{C^{k+\gamma}}\|g\|_{C^{k+\gamma}}$.

Homework 1.3. *Prove Theorem 1.2.*

Homework 1.4. *Prove Theorem 1.3.*

Homework 1.5. *Let* Ω *be a convex domain. Show that the norm of* $\|\cdot\|_{C^1(\bar{\Omega})}$ *and* $\|\cdot\|_{C^{0,1}(\bar{\Omega})}$ *are equivalent. Nevertheless,* $C^1(\bar{\Omega}) \subsetneq C^{0,1}(\bar{\Omega})$.

1.2.2 Compactness

Theorem 1.4. *Assume that* Ω *is bounded and* $k_1 + \gamma_1 < k_2 + \gamma_2$. *Then* $C^{k_1+\gamma_1}(\overline{\Omega}) \hookrightarrow\hookrightarrow C^{k_2+\gamma_2}(\overline{\Omega})$.

Compact subsets of Banach spaces Since compactness is important for the Schauder's fixed point theorem, here we provide two examples to demonstrate some subtleties in applications.

Example A. Let $\mathbf{X} = C(\bar{\Omega})$, where $\bar{\Omega} \subset\subset \mathbb{R}^n$ and $\partial\Omega$ is smooth. Let

$$B = \{u \in C^1(\bar{\Omega}); \|u\|_{C^1(\bar{\Omega})} \leq 3\}.$$

Question: Is B closed, and therefore compact, in $\mathbb{X}$?

Answer: It is true that B is bounded, convex, and precompact. But B is not closed, and therefore is not compact in $\mathbf{X}$.

Consider, for example, $\Omega = [-1, 1] \subset\subset \mathbb{R}$ and $f(x) = |x|$. Set $f_n(x) = \sqrt{\frac{1}{n} + x^2}$. Then for all $n \geq 1$, $f_n \in B$ and as $n \to \infty$, $f_n \to f$ in $\mathbf{X}$. Nevertheless, $f \notin B$.

Example B Now let Ω and $\mathbf{X}$ be as above and let $\alpha \in (0, 1]$ be arbitrarily fixed. Let

$$B = \{u \in C^{0+\alpha}(\bar{\Omega}) : \|u\|_{C^{0+\alpha}(\bar{\Omega})} \leq 3\}.$$

Question: Again, is B convex, closed, and compact in $\mathbb{X}$?

Answer: This time, the answer is yes. The verification is left as an exercise.

1.3 The Sobolev Spaces

1.3.1 Weak Derivatives

Definition 1.3. *Suppose* $u,\, v \in L^1_{loc}(\Omega)$ *and* $\alpha = (\alpha_1, \cdots, \alpha_n)$ *is a multi-index. We say that* v *is the* α^{th} *weak derivative of* u *and write* $v = D^\alpha u$ *if*

$$\int_\Omega u\, D^\alpha \phi\, dx = (-1)^{|\alpha|} \int_\Omega v\, \phi\, dx \qquad \forall \phi \in C_c^\infty(U).$$

Note that the order of differentiation is irrelevant. For example, $u_{x_i x_j} = u_{x_j x_i}$ if one of them exists.

Homework 1.6. *Show that a weak derivative, if it exists, is unique (up to a set of measure zero).*

Example 1.1. *Let $f(x) = |x|$, $x \in \mathbb{R}$. Then $f'(x) = 2H(x) - 1$, where $H(\cdot)$is the Heaviside function defined by*

$$H(x) = \begin{cases} 1 & \text{if } x > 0 \\ 1/2 & \text{if } x = 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Example 1.2. *The Heaviside function $H(\cdot)$ does not have a weak derivative. Indeed, $H'(x) = \delta(x)$ is the Dirac measure.*

1.3.2 The Definition

Definition 1.4. *Assume k is a non–negative integer and $p \in [1, \infty]$. The Sobolev space $W^{k,p}(\Omega)$ consists of those $L^p(\Omega)$ functions all of whose weak derivatives up to order k exist and are in $L^p(\Omega)$. Its norm is defined by*

$$\|f\|_{W^{k,p}(\Omega)} = \Sigma_{|\alpha| \le k} \|D^\alpha f\|_{L^p(\Omega)}$$

When $p \in [1, \infty)$, the space $W_0^{k,p}(\Omega)$ is the completion of $C_c^\infty(\Omega)$ under the $\|\cdot\|_{W^{k,p}(\Omega)}$ norm.

When $p = 2$, $W^{k,2}$ and $W_0^{k,2}$ is often written as H^k and H_0^k respectively, which are Hilbert spaces.

Theorem 1.5. *The Sobolev space $W^{k,p}(\Omega)$ is a Banach space.*

Homework 1.7. *Prove theorem 1.5.*

Homework 1.8. *Assume that Ω is bounded with smooth boundary. Show that $C^{0,1}(\bar{\Omega})$ and $W^{1,\infty}(\Omega)$ contain the same functions and their norms are equivalent.*

1.4 Certain Tools in Studying Sobolev Spaces

1.4.1 Mollification

Lemma 1.1. *Assume that $p \in [1, \infty)$. If $f \in L^p_{loc}(\Omega)$ then $p_\epsilon * f \to f$ in $L^p_{loc}(\Omega)$ as $\epsilon \to 0$.*

Homework 1.9. *Does the conclusion of Lemma 1.1 hold when $p = \infty$?*

1.4.2 Partition of Unity

Lemma 1.2. *Let Ω be an open set and $\Omega \subset \cup_{i=1}^\infty U_i$ where each U_i is compact. Then $\exists$ a function sequence $\{\phi_i(x)\}_{i=1}^\infty$ such that*

(i) $\Sigma_{i=1}^\infty \phi_i(x) = 1$ $\forall x \in \Omega$.

(ii) $\phi_i \in C_c^\infty(U_i)$.

(iii) $\forall x \in \Omega$, *there are only finitely many i such that $\phi_i(x) \neq 0$.*

1.4.3 Extension of Bounded Operators

Lemma 1.3. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two Banach Spaces and* $\mathbf{X}_1$ *be a subspace of* $\mathbf{X}$. *If* $\mathbf{T}$ *is a bounded linear operator from* $\mathbf{X}_1 \to \mathbf{Y}$, *then this is an extension of* $\mathbf{T}$ *from* $\mathbf{X} \to \mathbf{Y}$ *such that* $\|\mathbf{T}\|_{\mathbf{X}} \leq \|\mathbf{T}\|_{\mathbf{X}_1}$. *The extension is unique if* $\mathbf{X}_1$ *is dense in* $\mathbf{X}$.

Homework 1.10. *Prove Lemma 1.3*

1.5 Certain Properties of the Sobolev Spaces

1.5.1 Approximation

Theorem 1.6. *Assume that* Ω *is bounded,* $1 \leq p < \infty$ *and* $u \in W^{1,p}(\Omega)$. *Then* $\exists \{u_m\}_{m=1}^{\infty}$ *in* $C^{\infty}(\Omega) \cap W^{1,p}(\Omega)$ *such that* $u_m \to u$ *in* $W^{1,p}(\Omega)$. *In addition, if* $\partial\Omega \in C^1$ *them we can choose the sequence* $\{u_m\}$ *such that* $u_m \in C^{\infty}(\overline{\Omega})$ *for all* m.

Theorem 1.7. *Assume that* $p \in [1, \infty)$.

(a) $W^{k,p}(\mathbb{R}^n) = W_0^{k,p}(\mathbb{R}^n)$ *is the completion of* $C_c^{\infty}(\mathbb{R}^n)$ *under the* $\|\cdot\|_{W^{k,p}(\mathbb{R})}$ *norm.*

(b) If Ω *is bounded with* C^1 *boundary, then* $W^{k,p}(\Omega)$ *is the completion of* $C^{\infty}(\bar{\Omega})$ *under* $\|\cdot\|_{W^{k,p}(\Omega)}$ *norm.*

1.5.2 Extension

Definition 1.5. *Let* Ω_1 *and* Ω_2 *be two sets with* $\Omega_1 \subset \Omega_2$. *Let* u *be a function defined on* Ω_1 *and* v *be a function defined on* Ω_2. *If* $u \equiv v$ *on* Ω_1 *then we say* v *is an extension of* u.

Theorem 1.8. *Assume that* Ω *is bounded with* C^1 *boundary* $\partial\Omega$. *Let* V *be an open set such that* $\Omega \subset\subset V$ *(i.e.,* $\bar{\Omega} \subset V$*). Then there exists an operator* $E : W^{1,p}(\Omega) \to W_0^{1,p}(V)$ *such that*

(i) $Eu = u$ *on* Ω*;*

(ii) $\|Eu\|_{W^{1,p}(V)} \leq C(V, \Omega, p)\|u\|_{W^{1,p}(\Omega)}$.

Remark 1.1. *If* $u \in W_0^{1,p}(V)$, *then* $u \in W^{1,p}(\mathbb{R}^n)$ *if we extend* u *by zero outside of* V.

Homework 1.11. *Show that in Theorem 1.8, we can further require that (iii) Eu has compact support in* V.

1.5.3 Traces

Definition 1.6. *Assume that* Ω *is bounded with* C^1 *boundary* $\partial\Omega$. *Let* $f \in W^{1,1}(\Omega)$. *A function* $g \in L^1(\partial\Omega)$ *is called the trace of* f *on* $\partial\Omega$ *if*

$$\int_{\Omega} Df \cdot \phi dx = \int_{\partial\Omega} g\phi\vec{\nu} dS - \int_{\Omega} D\phi \cdot f dx \quad \forall \phi \in C^1(\overline{\Omega})$$

where $\vec{\nu}$ *is the exterior unit normal to* $\partial\Omega$. *Similarly let* Γ *be a* C^1 *portion of* $\partial\Omega$. *We say* $g = f|_{\Gamma}$ *if*

$$\int_{\Omega} D(\phi f) dx = \int_{\Gamma} \phi g \vec{\nu} dS$$

for all $\phi \in C^1(\overline{\Omega})$ *satisfying* $\phi = 0$ *on* $\partial\Omega\backslash\Gamma$.

Theorem 1.9. *Trace, if exists, is unique.*

Theorem 1.10. *Assume that* Ω *is bounded and has (piecewise)* C^1 *boundary* $\partial\Omega$. *Let* $p \in [1, \infty)$. *Then, for every* $f \in W^{1,p}(\Omega)$ *there is a trace, on* $\partial\Omega$. *In addition, denoting the trace by* $\mathbf{T}f$, *then*

$$\|\mathbf{T}f\|_{L^p(\partial\Omega)} \leq C\|f\|_{W^{1,p}(\Omega)}$$

where C *depends only on* p *and* Ω. *The operator* $\mathbf{T} : W^{1,p}(\Omega) \to \mathcal{L}^p(\partial\Omega)$ *is called the trace operator.*

Homework 1.12. *Show that for any* $p, q \geq 1$, *there does not exist a trace operator from* $L^p(\Omega)$ *to* $L^q(\partial\Omega)$.

Homework 1.13. *Assume that* Ω *is bounded with* C^1 *boundary* $\partial\Omega$. *Define a trace operator from* $C(\bar{\Omega})$ *to* $C(\partial\Omega)$.

1.6 Sobolev Embedding

1.6.1 Definition of Embedding

Definition 1.7. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be Banach spaces and* $\mathbf{X} \subset \mathbf{Y}$.

(i) We say that $\mathbf{X}$ *is embedded in* $\mathbf{Y}$, *and written as*

$$\mathbf{X} \hookrightarrow \mathbf{Y}$$

if there exists a constant C *such that* $\|u\|_{\mathbf{Y}} \leq C\|u\|_{\mathbf{X}}$ *for all* $u \in \mathbf{X}$.

(ii) We say that $\mathbf{X}$ *is compactly embedded in* $\mathbf{Y}$ *and written as*

$$\mathbf{X} \hookrightarrow\hookrightarrow \mathbf{Y}$$

if (a) $\mathbf{X} \hookrightarrow \mathbf{Y}$ *and (b) every bounded sequence in* $\mathbf{X}$ *is precompact in* $\mathbf{Y}$.

1.6.2 $W^{1,p}$ to L^q for $p < n$

Theorem 1.11. *[Gagliardo-Nirenberg-Sobolev Inequality] Assume that* $1 \leq p < n$. *Let*

$$p^* = \frac{np}{n-p}.$$

Then there exists $C(n, p)$ *such that for every* $u \in W^{1,p}(\mathbb{R}^n)$, $u \in L^{p^*}(\mathbb{R}^n)$ *and*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C(n, p)\|Du\|_{L^p(\mathbb{R}^n)}.$$

Corollary 1.12. *Let* n, p, p^* *be as in the previous theorem. Then for any open set* $\Omega \subset \mathbb{R}^n$ *and* $u \in W_0^{1,p}(\Omega)$,

$$\|u\|_{L^{p^*}(\Omega)} \leq C(n, p)\|DU\|_{L^p(\mathbb{R}^n)}.$$

Theorem 1.13. *Assume that $1 \le p < n$ and Let $p^* = \frac{np}{n-p}$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1 boundary. Then there exists a constant $C = C(n,p,\Omega)$ such that for every $u \in W^{1,p}(\Omega)$, $u \in L^{p^*}(\Omega)$ and*

$$\|u\|_{L^{p^*}(\Omega)} \le C\|u\|_{W^{1,p}(\Omega)}.$$

The key step in prove the above two theorems is the following lemma:

Lemma 1.4. *For any $u \in C_c^\infty(\mathbf{R^n})$*

$$\int_{\mathbf{R^n}} |u|^{\frac{n}{n-1}} dx \le \int_{\mathbf{R^n}} |Du| dx.$$

Homework 1.14. *Using Lemma 1.4, prove Theorems 1.11 and 1.13.*

1.6.3 $W^{1,p}$ to C^α for $p > n$

Theorem 1.14 (Morrey's Inequality). *Assume that $p > n$ and let*

$$\alpha = 1 - n/p.$$

Then for every $u \in C^1(\mathbb{R}^n)$,

$$\|u\|_{\alpha,\mathbb{R}^n} \le c(n,p)\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

where $c(n,p)$ depends only on n and p.

The theorem can be proven by utilizing the following Lemma. We denote by $B(x,r)$ the ball $\{y \in \mathbb{R}^n | |y-x| < r\}$.

Lemma 1.5. *There exists a constant $c(n)$ such that*

$$\frac{1}{|B(0,r)|}\int_{B(0,r)} |u(x) - u(z)| dz \le c(n) \int_{B(0,r)} \frac{|Du(z)|}{|x-z|^{n-1}} dz$$

for all $r \in (0,\infty)$, $u \in C^\infty(B(0,r))$, and $x \in B(0,r)$.

Theorem 1.15. *Let $p > n$ and $\alpha = 1 - n/p$.*

(1) For every $u \in W^{1,p}(\mathbb{R}^n)$, there exists a unique $u^* \in C^\alpha(\mathbb{R}^n)$ such that $u^* = u$ a.e. in $\mathbb{R}^n$ and

$$\|u^*\|_{C^\alpha(\mathbb{R}^n)} \le c(n,p)\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

(2) Let Ω be a bounded domain with C^1 boundary. Then for every $u \in W^{1,p}(\Omega)$ there exists a unique $u^* \in C^\alpha(\overline{\Omega})$ such that $u = u^*$ a.e. in Ω, and

$$\|u^*\|_{C^\alpha(\overline{\Omega})} \le c(p,n,\Omega)\|u\|_{W^{1,p}(\Omega)}.$$

Homework 1.15. *Prove Theorem 1.14 by using Lemma 1.5.*

Homework 1.16. *Prove Theorem 1.15 by using Theorem 1.14.*

1.6.4 General Sobolev Embedding

Theorem 1.16 (Sobolev Embedding). *Let Ω be a bounded domain with (piecewise) C^1 boundary. Let $m \geq 0$ and $k \geq 1$ be positive integers and $p \in [1, \infty)$.*

1. *Assume $kp < n$. Then*

   (a) $W^{m+k,p}(\Omega) \hookrightarrow W^{m,q}(\Omega)$ *where*

   $$q = \frac{np}{n - kp}.$$

   (b) $W^{m+k,p}(\Omega) \hookrightarrow\hookrightarrow W^{m,\hat{q}}(\Omega)$ *for every* $\hat{q} \in [1, \frac{np}{n-kp})$.

2. *Assume that $kp > n > (k-1)p$. Then*

   (a) $W^{m+k,p}(\Omega) \hookrightarrow C^{m+\alpha}(\bar{\Omega})$ *where*

   $$\alpha = k - n/p \in (0, 1).$$

   (b) $W^{m+k,p}(\Omega) \hookrightarrow\hookrightarrow C^{m+\gamma}(\Omega)$ *for every* $\gamma \in (0, k - n/p)$.

Homework 1.17. *Prove Lemma 1.16.*

Homework 1.18. *Prove Theorem 1.4.*

1.7 Functions Valued in Banach Spaces

In studying evolution equations, a space time function $u(x,t)$ can be simply viewed as a funtion from t to $\mathbf{u}(t) := u(\cdot, t)$ in certain Sobolev space $\mathbf{X}$, say $H^1(\Omega)$. That is, a linear evolution equation can be written as

$$\mathbf{u}_t = A\mathbf{u}$$

where A is a linear operator. Hence, one can pretty much regard the evolution equation as an ode and the solution $\mathbf{u}$ as a curve (trajectory) in certain Banach space. Indeed, a lot of the ode ideas can be used.

Here we provided the spaces needed in using this ideas.

Definition 1.8. *Let $\mathbf{X}$ be a Banach space, $T \in (0, \infty)$ and $p \in [1, \infty]$.*

1. *A function $\mathbf{u} : [0, T] \to \mathbf{X}$ is measurable if for every open set A in $\mathbf{X}$, the set $\{t \in [0, T]; \mathbf{u}(t) \in A\}$ is Lebesgue measurable.*

2. *The space $L^p(0, T; \mathbf{X})$ consits of all measurable functions $\mathbf{u} : [0, T] \to \mathbf{X}$ such that*

   $$\|\mathbf{u}\|_{L^p(0,T;X)} := \left(\int_0^T \|\mathbf{u}(t)\|^p \, dt \right)^{1/p} < \infty \quad \text{if } p \in [1, \infty)$$

   and

   $$\|\mathbf{u}\|_{L^\infty(0,T;X)} = ess \sup_{0 \leq t \leq T} \|\mathbf{u}(t)\| < \infty \quad \text{if } p = \infty.$$

3. *The space* $C([0,T];\mathbf{X})$ *consists of all continuous functions* $\mathbf{u}:[0,T]\to\mathbf{X}$ *with*

$$\|\mathbf{u}\|_{C([0,T];X)} := \sup_{0\le t\le T}\|\mathbf{u}(t)\| < \infty.$$

4. *For* $\gamma\in(0,1]$*, The Hölder space* $C^\gamma(0,T;\mathbf{X})$ *consists of all continuous functions* $\mathbf{u}:[0,T]\to\mathbf{X}$ *such that*

$$\|\mathbf{u}\|_{C^\gamma(0,T;\mathbf{X})} := \sup_{0\le t<T}\|\mathbf{u}(t)\|_{\mathbf{X}} + \sup_{0\le\tau<t<T}\frac{\|\mathbf{u}(t)-\mathbf{u}(\tau)\|_{\mathbf{X}}}{|t-\tau|^\gamma} < \infty.$$

5. *If* $\mathbf{u},\mathbf{v}\in L^1(0,T;\mathbf{X})$*, we shall say that* $\mathbf{v}$ *is the (weak) derivative of* $\mathbf{u}$*, written as* $\mathbf{u}'=\mathbf{v}$*, if*

$$\int_0^T \phi'(t)\mathbf{u}(t)dt = -\int_0^T \phi(t)\mathbf{v}(t)dt$$

for all scalar test functions $\phi\in C_c^\infty(0,T;\mathbb{R})$*.*

6. *The Sobolev space* $W^{1,p}(0,T;\mathbf{X})$ *consists of all functions in* $L^p(0,T;\mathbf{X})$ *whose weak first derivatives also lie in* $L^p(0,T;\mathbf{X})$*.* $W^{1,p}(0,T;\mathbf{X})$ *carries the evident norm.*

When $\gamma=1$, the resulting Hölder space in item 4 will be denoted by $C^{0,1}(0,T;\mathbf{X})$, since $C^1(0,T;\mathbf{X})$ has its obvious meaning.

Theorem 1.17. *Let* $\mathbf{X}$ *be a Banach space and* $T\in(0,\infty)$*.*

(1) If $\mathbf{u}\in W^{1,1}(0,T;\mathbf{X})$*, then* $\mathbf{u}\in C(0,T;\mathbf{X})$ *and*

$$\mathbf{u}(t)=\mathbf{u}(s)+\int_s^t \mathbf{u}'(\tau)d\tau \quad \forall t,s\in[0,T].$$

(2) For all $p\in[1,\infty)$*,* $W^{1,p}(0,T;\mathbf{X})\hookrightarrow C^{1-1/p}(0,T;\mathbf{X})$*.*

(3) The space $W^{1,\infty}(0,T;\mathbf{X})$ *and* $C^{0+1}(0,T;\mathbf{X})$ *are the same.*

Idea of the Proof. When $p\in[1,\infty)$, one first proves the Embedding for smooth functions, and then use the extension of bounded linear operators over Banach spaces. For $p=\infty$, one needs to use the definition of weak derivatives. □

In study evolutions, it is quite often that $\mathbf{u}(t)\in L^p(0,T;\mathbf{X}_1)$ and $u'\in L^q(0,T;\mathbf{X}_2)$ where $\mathbf{X}_1\hookrightarrow\hookrightarrow\mathbf{X}_2$. In this case, one can obtain special regularity of $\mathbf{u}$ in some intermediate Banach space $\mathbf{X}_3$ where $\mathbf{X}_1\hookrightarrow\hookrightarrow\mathbf{X}_3\hookrightarrow\hookrightarrow\mathbf{X}_2$. Here we just give one example, for the sake of our applications.

Theorem 1.18. *Let* $\mathbf{u}\in L^2(0,T;H_0^1(\Omega))$ *and* $\mathbf{u}'\in L^2(0,T;H^{-1}(\Omega))$*. Then* $\mathbf{u}\in C([0,T];L^2(\Omega))$ *(modulo sets of measure zero). The mapping* $t\mapsto\|\mathbf{u}(t)\|^2_{L^2(\Omega)}$ *is absolutely continuous, with*

$$\frac{d}{dt}\|\mathbf{u}(t)\|^2_{L^2(\Omega)} = 2\langle\mathbf{u}'(t),\mathbf{u}(t)\rangle.$$

Furthermore,

$$\max_{0\le t\le T}\|\mathbf{u}(t)\|_{L^2(\Omega)} \le C(\|\mathbf{u}\|_{L^2(0,T;H_0^1)}+\|\mathbf{u}'\|_{L^2(0,T;H^{-1})}).$$

Proof[Proof sketch] The trick is to consider two mollified $\mathbf{u}$, namely $\mathbf{u}^\epsilon$, $\mathbf{u}^\delta$ in an effort to prove that the family $\{\mathbf{u}^\epsilon\}_{\epsilon>0}$ is uniformly Cauchy in $C([0,T];L^2(\Omega))$ as $\epsilon \to 0$. Note that indeed each mollified function is in $C(0,T;L^2(\Omega))$. Pick some point $s \in (0,T)$ such that

$$\mathbf{u}^\epsilon(s) \to \mathbf{u}(s) \quad \text{in } L^2(\Omega),$$

and then consider the zeroth order Taylor expansion for $\left\|\mathbf{u}^\epsilon(t) - \mathbf{u}^\delta(t)\right\|_{L^2}^2$ about $t = s$. Using this expansion, one verifies immediately that $\mathbf{u}^\epsilon$ is uniformly Cauchy with respect to ϵ, and so the mollified functions tend to a limit in $C([0,T];L^2(\Omega))$. Moreover, it is trivial that this limit must be equal a.e. to $\mathbf{u}$. Hence the first assertion of the theorem is true.

The second assertion may now proven by showing that it holds for $\mathbf{u}^\epsilon$, and then passing to the limit as $\epsilon \to 0$. And the third assertion follows by the estimate $|\langle \mathbf{u}', \mathbf{u}\rangle| \le \|\mathbf{u}'\|_{H^{-1}} \|\mathbf{u}\|_{H_0^1}$.

Chapter 2

SECOND ORDER ELLIPTIC EQUATIONS

In this chapter, we study the equation

$$\mathcal{L}u = f \tag{2.1}$$

where $\mathcal{L}$ denotes a second order partial differential operator having either the *divergence form*

$$\mathcal{L}u = -(a^{ij}u_{x_i} + d^j u)_{x_j} + b^i u_{x_i} + cu \tag{2.2}$$

or else *non-divergence form*

$$\mathcal{L}u = -a^{ij}u_{x_i x_j} + b^i u_{x_i} + cu. \tag{2.3}$$

Here a^{ij}, b^j, d^i and c are functions of $x \in \Omega \subset \mathbb{R}^n$. Here we omit the summation from 1 to n sign whenever there is a repeating super-index and sub-index.

2.1 Ellipticity and Weak Solutions

Definition 2.1. *Let $\mathcal{L}$ be defined as (2.2) or (2.3).*

(1) We say that $\mathcal{L}$ is elliptic in Ω if

$$a^{ij}(x)\xi_i\xi_j > 0 \quad \forall x \in \Omega, \xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n \setminus \{0\}.$$

(2) We say that L is uniformly elliptic if there exists a constant $\theta > 0$ such that

$$a^{ij}(x)\xi_i\xi_j \geq \theta \|\xi\|^2 \quad \forall x \in \Omega, \xi \in \mathbb{R}^n.$$

When $\mathcal{L}$ is in the non–divergence form (2.3), we can assume, without loss generality, that $a^{ij} = a^{ji}$. Then the ellipticity means that the matrix $(a^{ij})_{n\times n}$ is positive definite.

Definition 2.2. *The bilinear form $B[\cdot,\cdot]$ associated with $\mathcal{L}$ in (2.3) is*

$$B[u,v] = \int_\Omega \left\{ a^{ij}u_{x_i}v_{x_j} + d^j u v_{x_j} + b^i u_{x_i} v + cuv \right\} dx.$$

Definition 2.3. *Let $\mathcal{L}$ be as in (2.3). We say that u is a weak solution to*

$$\begin{cases} \mathcal{L}u = F := f - f^i_{x_i} & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{2.4}$$

if $u - g \in H^1_0(\Omega)$ and

$$B[u,v] = \langle F, v\rangle := \int_\Omega \{fv + f^i v_{x_i}\} \quad \forall\, v \in H^1_0(\Omega).$$

Remark 2.1. *(1) For the definition to make sense, it is typically required that the coefficients of $\mathcal{L}$ satisfying the following:*

$$a^{ij} \in L^\infty(\Omega), \qquad b^i, d^i \in L^n(\Omega), \qquad c \in L^n(\Omega). \tag{2.5}$$

(When $n = 2$, we require $b^i, d^i \in L^p$ and $c \in L^{p/2}$ for some $p > n$.) These conditions make sure that all $a^{ij}u_{x_i}v_{x_j}$, $b^i u_{x_i} v$, $d^j u v_{x_j}$, and cuv are in $L^1(\Omega)$ whenever $u, v \in H^1(\Omega)$.

(2) Also, one needs $F \in H^{-1}$, the space of all bounded linear functional of $H^1_0(\Omega)$; in particular,

$$f \in L^{(n+2)/(2n)}(\Omega), \qquad f_i \in L^2(\Omega).$$

(3) It is necessary to assume that g, need only be defined on $\partial\Omega$, has an extesion, still denoted by g, over $H^1(\Omega)$.

Homework 2.1. *Assume (2.5). Show that $B[u,v]$ is well–defined for all $u, v \in H^1(\Omega)$.*

2.2 Some Tools for Weak Solutions

2.2.1 The Lax-Milgram Theorem

We shall employ the following theorem to prove the existence of weak solutions to (2.4).

Theorem 2.1 (Lax-Milgram). *Let $\mathbf{H}$ be a Hilbert space and $\mathbf{H}^*$ be its dual. Denote by $\|u\|$ the norm of $u \in \mathbf{H}$ and by $\langle F, u\rangle$ the value of the functional $F \in \mathbf{H}^*$ at $u \in \mathbf{H}$. Assume that $L : \mathbf{H} \to \mathbf{H}^*$ is a linear operator having the following properties:*

1. *L is bounded; that is, there exists $\alpha > 0$ such that*

$$|\langle Lu, v\rangle| \le \alpha\|u\|\,\|v\| \quad \forall u, v \in \mathbf{H};$$

2. *L is coercive; that is, there eixts $\beta > 0$ such that*

$$\langle Lu, u\rangle \ge \beta\|u\|^2 \quad \forall u \in \mathbf{H}.$$

Then for every $F \in H^*$, there exists a unique $u \in \mathbf{H}$ such that $Lu = F$, i.e.

$$\langle Lu, v\rangle = \langle F, v\rangle \quad \forall v \in \mathbf{H}.$$

Later on, we shall show that for some large λ, $\mathcal{L} + \lambda$ satisfies the Lax–Milgram theorem. For other values of λ, we need to use the Fredholm alternative.

2.2.2 The Fredholm Alternative

Recall that an operator K is *compact* if the image of every bounded set is precompact.

Theorem 2.2. *Let* $\mathbf{H}$ *be a Hilbert space and* $K : \mathbf{H} \to \mathbf{H}$ *be a linear compact operator. Then the following holds:*

1. *The kernel* $Ker(\mathbf{I} - K) := \{u \in \mathbf{H}; u = Ku\}$ *is finite dimensional;*

2. *The range* $R(\mathbf{I} - K) := \{u - Ku; u \in \mathbf{H}\}$ *is closed and* $R(\mathbf{I} - K) = Ker(\mathbf{I} - K)^{\perp}$*;*

3. $Ker(\mathbf{I} - K) = \{0\}$ *if and only if* $R(\mathbf{I} - K) = \mathbf{H}$*;*

4. *dim* $Ker(\mathbf{I} - K) =$*dim* $Ker(\mathbf{I} - K^*)$.

Corollary 2.3 (Fredhold Alternative). *Assume the consitions as in the previous theorem. Then one of the following alternative holds:*

$$(\alpha): \quad \forall f \in H,\ \exists\,!\, u \in H \ \ni u - Ku = f;$$
$$(\beta): \quad \exists\, u \in \mathbf{H} \ \ni u = Ku.$$

In addition, should (β) *hold,*

$$(\gamma) : u - Ku = f \text{ has a solution} \Leftrightarrow f \in \mathrm{Ker}(\mathbf{I} - K^*)^{\perp}.$$

More general result can be stated as follows:

Theorem 2.4. *Let* K *be a linear compact operator from a Hilbert Space* $\mathbf{H}$ *into itself. Define the (point) spectrum of* K *by*

$$\sigma_p(K) = \{\lambda \in \mathbb{C}; \exists\, x \neq 0 \ni Kx = \lambda x\}.$$

Then

1. $\sigma_p(K)$ *is bounded, and for each* $\lambda \in \sigma_p(K) \setminus \{0\}$*, the dimension of the kernel of* $\lambda\mathbf{I} - K$ *is finite;*

2. *The only accumulation point of* $\sigma_p(K)$ *is the origin.*

Theorem 2.5 (Fredholm Alternative). *Assume that* $\mathbf{H}$ *is a Hilbert space and* $K : H \to H$ *is a linear compact operator. Let* $\lambda \in \mathbb{C} \setminus \{0\}$ *be any fixed number. Then* $R(\lambda\mathbf{I} - K) = \mathbf{H} \Leftrightarrow Ker(\lambda\mathbf{I} - K) = \{0\}$. *Also, one of the following holds:*

(i) *if* $\lambda \notin \sigma_p(K)$*, then* $(\lambda I - K)^{-1}$ *exists and is bounded;*

(ii) *if* $\lambda \in \sigma_p(K)$*, then* $(\lambda I - K)u = f$ *is solvable iff* $f \in (N^*)^{\perp}$*, where* $N^* = \left\{v \in \mathbf{H}; (\overline{\lambda} I - K^*)v = 0\right\}$.

2.3 Existence of Weak Solutions

Theorem 2.6. *Let $\mathcal{L}$ be as in (2.3). Assume that $\mathcal{L}$ is uniformly elliptic and that (2.5) holds. Then there exists $\mu \in \mathbb{R}$ such that for every $\lambda > \mu$, $g \in H^1(\Omega)$ and $F \in H^{-1}(\Omega)$, the problem*

$$\begin{cases} Lu + \lambda u = F \text{ in } H^{-1}(\Omega) \\ u - g \in H_0^1 \end{cases} \tag{2.6}$$

has a unique solution. In addition, there exists a positive constant C, which depends only on λ, Ω, and the coefficients of $\mathcal{L}$, such that

$$\|u\|_{H^1(\Omega)} \le C(\|g\|_{H^1(\Omega)} + \|F\|_{H^{-1}(\Omega)}).$$

Idea of the Proof.

Step 1. By working on $v = u - g$, we can assume that $g \equiv 0$. We let $\mathbf{H} = H_0^1(\Omega)$ be our working space. Note that we can define $(u, v) = \int_\Omega Du \cdot Dv\, dx$ as the inner product of $\mathbf{H}$. Also, $\mathbf{H}^* = H^{-1}(\Omega)$.

Define $L^\lambda : H_0^1(\Omega) \to H^{-1}$ by

$$\langle L^\lambda u, v\rangle = B[u, v] + \lambda(u, v)_{L^2}.$$

Then (2.6) is equivalent to $L^\lambda u = F$ in $H^{-1}(\Omega) = \mathbf{H}^*$.

Step 2. Using condition (2.5) and Sobolev imbedding, one can show that for any real $\lambda \in \mathbb{R}$, L^λ is bounded.

Step 3. It remains to show that L^λ is coercive, for appropriate λ. Indeed, we show that for some $\mu > 0$,

$$B[u, u] \ge \frac{\theta}{2}\|u\|_{H_0^1(\Omega)} - \mu\|u\|^2_{L^2(\Omega)} \quad \forall u \in H_0^1(\Omega). \tag{2.7}$$

This estimate implies that L^λ is coercive provided that $\lambda \ge \mu$. The assertion of the theorem then follows from the Lax-Milgram theorem.

To show (2.7), we first use the ellipticity condition to obtain

$$\int_\Omega a^{ij} u_{x_i} u_{x_j}\, dx \ge \theta \int_\Omega |Du|^2\, dx = \theta\|u\|_H^2.$$

To estimate integrals $\int_\Omega \{(b^i + d^i)uu_{x_i} + cu^2\}\, dx$, we use the following decomposition result:

Lemma 2.1. *Let $q \in [1, \infty)$ and $f \in L^q(\Omega)$. Then for every $\epsilon > 0$, there exists $f_1^\epsilon \in L^\infty(\Omega)$ and $f_2^\epsilon(\Omega) \in L^q(\Omega)$ such that $f = f_1^\epsilon + f_2^\epsilon$ and $\|f_2^\epsilon\|_{L^q(\Omega)} \le \epsilon$.*

Utilizing the decomposition, (2.7) then follows from the following inequalities: (For simplicity, assume that $n > 2$):

$$\begin{aligned} \int_\Omega (|f|\ |Du|\ |u| + |g|u^2)\, dx &\le \|f\|_{L^\infty}\|Du\|_{L^2}\|u\|_{L^2} + \|g\|_{L^\infty}\|u\|^2 \\ &\le \delta\|Du\|^2_{L^2} + \left(\|g\|_{L^\infty} + \frac{\|f\|^2_{L^\infty}}{4\delta}\right)\|u\|^2_{L^2}, \\ \int_\Omega (|f||Du||u| + |g|u^2)\, dx &\le \|f\|_{L^n}\|Du\|_{L^2}\|u\|_{L^{2^*}} + \|g\|_{L^{n/2}}\|u\|^2_{L^{2^*}} \\ &\le C(n, \Omega)(\|f\|_{L^n} + \|g\|_{L^{n/2}})\|Du\|^2_{L^2} \end{aligned}$$

by the Sobolev embedding. Here $2^* = 2n/(n-2)$. (When $n = 2$, an obvious modification is needed.) □

Note that problem (2.6) can be written as

$$\begin{aligned} L^\lambda u = F \quad &\Leftrightarrow \quad (L^\mu + (\lambda - \mu))u = F \text{ in } \mathbf{H}^* \\ &\Leftrightarrow \quad \{\mathbf{I} + (\lambda - \mu)(L^\mu)^{-1} I\}u = (L^\mu)^{-1}F \text{ in } \mathbf{H} \\ &\Leftrightarrow \quad (\omega \mathbf{I} + K)u = G \text{ in } \mathbf{H} \end{aligned}$$

where $\omega = 1/(\lambda - \mu), K = (L^\mu)^{-1}I$, I is the embedding from H_0^1 to H^{-1}, and $G = \omega(L^\mu)^{-1}F$. Since the embedding $I : H_0^1 \to H^{-1}$ is compact and $(L^\mu)^{-1} : H^{-1} \to H_0^1$ is bounded, we see that K is a compact operator from H_0^1 to itself. After applying the Fredholm alternative and a simple calculation, one then can prove the following:

Theorem 2.7. *Assume the conditions of Theorem 2.6. Then the following holds:*

1. *For every $\lambda \in \mathbb{C}$, $(\mathcal{L} + \lambda)^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)$ exists and is bounded if and only if λ is not an eigenvalue of $\mathcal{L}$, i.e., if and only if $(\mathcal{L} + \lambda)u = 0$ has only the trivial solution in $H_0^1(\Omega)$.*
2. *If λ is an eigenvalue of $\mathcal{L}$, i.e., there exists $u \in H_0^1(\Omega)$ such that $(\mathcal{L} + \lambda)u = 0$, then*

   (a) The dimension of the eigenspace of $\mathcal{L}$ with respect to λ is finite,

   (b) $(\mathcal{L} + \lambda)u = F$ is solvable if and only if $F \perp Ker(\mathcal{L}^* + \bar{\lambda})$, where

   $$\mathcal{L}^* u = -(a^{ij}u_{x_j} + b^i u)_{x_i} + d^j u_{x_j} + cu.$$
3. *We can order all the eigenvalues of $\mathcal{L}$ by $\lambda_1, \lambda_2, \dots$ with*

   (a) $\Re(\lambda_1) \le \Re(\lambda_2) \le \dots$;

   (b) $\lim_{k\to\infty} \Re(\lambda_k) = \infty$.

2.4 H^m Regularity

For simplicity, we study the regularity of solution to

$$\mathcal{L}u := -(a^{ij}u_{x_i})_{x_j} = f \quad \text{in } H^{-1}(\Omega) \tag{2.8}$$

Theorem 2.8. *Assume $\mathcal{L}$ is uniformly ellpitic.*

(1) If $a^{ij} \in L^\infty$ then for each $f \in H^{-1}(\Omega)$, (2.8) together with (boundary condition) $u \in H_0^1(\Omega)$ admits a unique solution and the solution satisfies

$$\|u\|_{H_0^1(\Omega)} \le C\|f\|_{H^{-1}(\Omega)}$$

where C depends only on the ellipticity constant of $\mathcal{L}$.

(2) Assume that $u \in H^1(\Omega)$ is a weak solution to (2.8) and $m \geq 1$ is a non–negative integer such that

$$a^{ij} \in C^{m+1}(\Omega), \quad f \in H^m(\Omega)$$

Then for each $V \subset\subset \Omega$, $u \in H^{m+1}(V)$ and there exists $C = C(a^{ij}, V)$ such that

$$\|u\|_{H^{m+2}(V)} \leq C(\|f\|_{H^m(\Omega)} + \|u\|_{L^2(\Omega)}).$$

(3) Assume that $\partial\Omega \in C^{m+2}$, and that

$$a^{ij} \in C^{m+1}(\bar{\Omega}), f \in H^m(\Omega).$$

Then the unique solution $u \in H_0^1(\Omega)$ to (2.8) is in $H^{m+1}(\Omega)$ and there exists a constant $C = C(a^{ij}, \Omega)$ such that

$$\|u\|_{H^{m+2}(\Omega)} \leq C\|f\|_{H^m(\Omega)}.$$

Homework 2.2. *Extend the regularity theorem for $\mathcal{L}$ defined in (2.2).*

2.5 An L^∞ Estimate

In this section we consider the operator

$$Lu = -(a^{ij}u_{x_i} + d^j u)_{x_j} + b^i u_{x_i} + cu. \tag{2.9}$$

We assume the following:

(1) There exist positive constants λ $\wedge$ such that $\lambda|\xi|^2 \leq a^{ij}\xi_i\xi_j \leq \wedge|\xi|^2$ for all $\xi \in \mathbb{R}^n$;

(2) $a^{ij} \in L^\infty, d^j,\ b_i \in L^n,\ c \in L^{n/2}$;

(3) $c - (d^j)_{x_j} \geq 0$ in the distributional sense;

(4) Ω is bounded and $\partial\Omega$ satisfies the exterior cone conditions.

Definition 2.4. *w is called a sub (sup) solution to $\mathcal{L}u = F$ if $Lw \leq F$ ($Lw \geq F$) in the distribution sense.*

Theorem 2.9 (L^∞ estimate). *Assume (1)–(4). Let u be such that $Lu \leq f - (f_i)_{x_i}$ where $f \in L^{\frac{np}{n+p}}$ and $f_i \in L^p$ for some $p > n$. Then*

$$ess \sup_\Omega u \leq ess \sup_{\partial\Omega} u + C\ \{\|f\|_{\frac{np}{n+p}} + \sigma\|f\|_p\ \}\ |\Omega|^{\frac{1}{n}-\frac{1}{p}}$$

where C is a constant depending on $\lambda, \wedge, d^j, b^i, c, \Omega$, but not on the lower bound on $|\Omega|$.

Corollary 2.10 (Weak maximum principle). *Assume (1)–(4). Also assume that $\mathcal{L}u \leq 0$. Then*

$$ess \sup_\Omega u \leq ess \sup_{\partial\Omega} u.$$

2.6 The Hölder Estimate

We consider, for simplicity,

$$Lu := -(a^{ij}u_{x_i})_{x_j} \tag{2.10}$$

We assume that

$$|a^{ij}| \le \wedge, \quad a^{ij}\xi_i\xi_j \ge \lambda|\xi|^2.$$

2.6.1 Some Auxillary Lemmas

Lemma 2.2. *Assume that ϕ is a nonnegative, nonincreasing function defined on $[0,\infty)$, and satisfies, for some constants $c>0$, $\alpha>0$ and $\beta>1$,*

$$(h-k)^\alpha\phi(h) \le C\phi^\beta(k) \quad \forall\ 0 \le k \le h.$$

Then $\phi(h)=0$ for all $h \ge d$ where

$$d = 2^{\frac{\beta}{\beta-1}}C^{\frac{1}{\alpha}}\phi^{\frac{\beta-1}{\alpha}}(0).$$

Lemma 2.3. *Suppose ϕ is defined on $[T_0,T_1]$ with $0 \le T_0 < T_1$ and satisfies for some $\theta \in (0,1)$, $A>0$ and $B>0$,*

$$\phi(s) \le \theta\phi(t) + A(t-s)^{-\alpha} + B \quad \textit{for all} \ \ T_0 \le s < t \le T_1.$$

Then for some $C=C(\theta)$,

$$\phi(s) \le C\ \{A(t-s)^{-\alpha}+B\ \} \quad \forall\ T_0 \le s < t \le T_1.$$

Lemma 2.4. *Assume that $\Phi \in C^{0,1}(\mathbb{R};\mathbb{R})$ is convex. Suppose that either (i) u is a weak solution of $Lu=0$ or (ii) u is a weak subsolution (i.e., $Lu \le 0$) and $\phi'(s) \le 0$. Then $\Phi(u)$ is a supersolution.*

Lemma 2.5. *Suppose ω is a nonnegative, nondecreasing function defined on $[0,R_0]$ and satisfies, for some constant $\theta \in (0,1), \eta \in (0,1)$ and $K>0$,*

$$\omega(\theta R) \le \eta\omega(R) + KR^\gamma \quad \forall\ R \in (0,R_0].$$

Then there exist positive constants ν and C, which depend only on θ and η such that

$$\frac{\omega(r)}{r^\nu} \le \frac{C(\omega(R_0)+KR_0^\gamma)}{R_0^\nu} \quad \forall r \in (0,R_0].$$

2.6.2 Maximum of Subsolutions

In the sequel, $u \vee M = \max\{u, M\}, u \wedge m = \min\{u, m\}$, and B_R is a ball of radius R centerer at the origin.

Lemma 2.6 (Interior). *Assume that $u \in H^1(B_R)$ is a subsolution to (2.10). Then for any $p > 0$ and $\theta \in (0, 1)$, there exists $C = C(\wedge/\lambda, \theta, p, n)$ such that*

$$ess \sup_{B_{\theta R}} (u \vee 0)^p \le C R^{-n} \int_{B_R} (u \vee 0)^p,$$

Lemma 2.7 (Boundary). *Assume that u is a subsolution in Ω and $0 \in \partial\Omega$. Let $M = \sup_{\partial\Omega \cap B_R} u$. Then for any $p > 0$, there exists $C = C(p, \Omega, \wedge/\lambda)$ such that*

$$\sup_{B_{\theta_R} \cap \Omega} (u \vee M)^p \le C R^{-n} \left(\int_{B_R \cap \Omega} (u \vee M)^p + \int_{B_R \setminus \Omega} M^p \right).$$

2.6.3 Minimum of Positive Supersolutions

Lemma 2.8 (Interior). *Suppose $Lu \ge 0$ and $u \ge 0$ in B_R. Then for any $\theta \in (0, 1)$,*

$$ess \inf_{B_{\theta_R}} u^p \ge C(\theta, \wedge/\lambda, n) R^{-n} \int_{B_R} u^p$$

for some p depending on $n, \wedge/\lambda$.

Lemma 2.9 (Boundary). *Assume $Lu \ge 0$ and $u \ge 0$ in Ω, $\partial\Omega$ satisfies the exterior cone condition, and $0 \in \partial\Omega$. Let $m = ess \inf_{B_R \cap \Omega} v$. Then for any $\theta \in (0, 1)$, there exist positive $c = c(\Omega, \wedge/\lambda)$ and $p = p(\wedge/\lambda, \Omega, \theta)$ such that*

$$\inf_{B_{\theta R} \cap \Omega} (u \wedge m)^p \ge c R^{-n} \left(\int_{B_R \cap \Omega} (v \wedge m)^p + \int_{B_R \setminus \Omega} m^p \right)$$

2.6.4 The Harnack Inequality

Theorem 2.11. *Assume that $Lu = 0$ and $u \ge 0$ in B_R. Then for each $\theta \in (0, 1)$, there exists a positive constant $C = C(\wedge/\lambda, \theta)$ such that*

$$ess \sup_{B_{\theta R}} u \le C \;\; ess \inf_{B_{\theta R}} u.$$

2.6.5 The Hölder Estimate

Theorem 2.12 (Interior Hölder Estimate). *Assume that $Lu = f - f^i_{x_i}$ in B_R and for some $p > n$, $f^i \in L^p$ and $f \in L^{np/(n+p)}$. Then there exist positive constant $C = C(\wedge, \lambda, n, p)$ and $\alpha = \alpha(\wedge/\lambda, n, p) \in (0, 1)$ such that*

$$\frac{\omega(\rho)}{\rho^\alpha} \le \frac{C}{R^\alpha} \Big\{ \omega(R) + R^{\frac{1}{n} - \frac{1}{p}} (\|f\|_{p, B_R} + \Sigma_i \|f^i\|^i_{q, B_R}) \Big\} \quad \forall \rho \in (0, R].$$

where $\omega(\rho) = ess \sup_{B_\rho} u - ess \inf_{B_\rho} u$ is the oscilation of u over B_ρ.

Theorem 2.13 (Global Hölder Estimate). *Assume that $\partial\Omega$ satisfies the exterior cone condition and $Lu = f - f^i_{x_i}$ with $f \in L^{pn/(n+p)}$ and $f_i \in L^p$ for some $p > n$. Then there exists $C(\wedge, \lambda, p, \Omega)$ and $\alpha(\wedge/\lambda, \Omega, p) \in (0,1)$ such that*

$$\|u\|_{C^\alpha(\overline{\Omega})} \leq C \left\{ \|u\|_{C^\alpha(\partial\Omega)} + \Sigma_i \|f^i\|_{p,\Omega} + \|f\|_{\frac{np}{n+p},\Omega} \right\}$$

Remark 2.2. *The estimate extends to general $\mathcal{L}$, if we allow the resulting constant C to also depend on $\max\{1, R\}$.*

2.7 The L^p Estimate

Theorem 2.14 (Interior and Global L^p Estimate). *Let*

$$\mathcal{L}u = -a^{ij}u_{x_i x_j} + b^i u_{x_i} + cu.$$

Assume that

(i) there exists $\lambda > 0$ such that $a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2$ for all $\xi \in \mathbb{R}^n$ and $x \in \Omega$;

(ii) $a^{ij} \in C^0(\overline{\Omega})$, $b^i \in L^\infty$, and $c \in L^\infty$;

(iii) For some $p \in (1,\infty)$, $\mathcal{L}u \in L^p(\Omega)$.

Then $u \in W^{2,p}_{loc}(\Omega)$ and for any $\hat{\Omega} \subset\subset \Omega$,

$$\|u\|_{W^{2,p}(\hat{\Omega})} \leq C(\hat{\Omega})(\|\mathcal{L}u\|_{p,\Omega} + \|u\|_{p,\Omega}). \tag{2.11}$$

If Ω is bounded and $\partial\Omega \in C^2$ and $u = 0$ on $\partial\Omega$, then (2.11) holds with $\hat{\Omega} = \Omega$.

2.8 The Schauder Estimate

Theorem 2.15 (The Interior and Global Schauder estimate). *Let $\alpha \in (0,1)$ and*

$$\mathcal{L}u = -a^{ij}u_{x_i x_j} + b^i u_{x_i} + cu.$$

Assume that

(i) $\mathcal{L}$ is uniformly elliptic;

(ii) $a^{ij}, b^i, c \in C^\alpha(\overline{\Omega})$;

(iii) $\mathcal{L}u \in C^\alpha(\Omega)$.

Then $u \in C^{2+\alpha}_{loc}(\Omega)$ and for any $\hat{\Omega} \subset\subset \Omega$,

$$\|u\|_{C^{2+\alpha}(\hat{\Omega})} \leq C(\hat{\Omega}) \left\{ \|\mathcal{L}u\|_{C^\alpha(\Omega)} + \|u\|_{C^0(\Omega)} \right\}. \tag{2.12}$$

If Ω is bounded and $\partial\Omega \in C^{2+\alpha}$ and $u = 0$ on $\partial\Omega$, then (2.12) holds with $\hat{\Omega} = \Omega$.

Chapter 3

SECOND ORDER EVOLUTION EQUATIONS

In this chapter, we study evolution equation, of parabolic type

$$u_t + Lu = f \tag{3.1}$$

and of hyperbolic type

$$u_{tt} + Lu = f \tag{3.2}$$

where L is a second order elliptic operator, in the divergence form

$$Lu = -(a^{ij}u_{x_i} + d^j u)_{x_j} + b^i u_{x_i} + cu \tag{3.3}$$

or in the non–divergence form

$$Lu = -a^{ij}u_{x_i x_j} + b^i u_{x_i} + cu. \tag{3.4}$$

We notice that if we write $\mathbf{u} = (u_1, u_2) := (u, u_t)$, then the hyperbolic equation (3.2) can be written as a first order (in time) evolution

$$\mathbf{u}_t = \mathcal{L}\mathbf{u} \tag{3.5}$$

where $\mathcal{L}\mathbf{u} = (u_2, Lu_1)$.

Under approriate setting, (3.5) can be regarded as an ODE.

In studying evolution equations, a space time function $u(x,t)$ can be simply viewed as a funtion from t to $\mathbf{u}(t) := u(\cdot, t)$ in certain Sobolev space $\mathbf{X}$, say $H^1(\Omega)$. That is, one can regard an evolution equation as an "ODE", and the solution $\mathbf{u}$ as a curve (trajectory) in certain Banach space.

3.1 Parabolic Equations in Divergence Form

We shall study the following initial value problem in the cylinder $\Omega_T = \Omega \times (0,T]$, where Ω is a bounded domain in $\mathbb{R}^n$:

$$\begin{cases} u_t + Lu = f & \text{in } \Omega_T \\ u = 0 & \text{in } \partial\Omega \times [0,T] \\ u = g & \text{in } \Omega \times \{t=0\} \end{cases} \tag{3.6}$$

Here f and g are given functions, and $L = L(x,t)$ is a uniformly elliptic differential operator in the divergence form

$$Lu = -(a^{ij}u_{x_i} + d^j u)_{x_j} + b^i u_{x_i} + cu \tag{3.7}$$

Parallel to that of ellpitic case, for (a.e.) $t \in [0,T]$, we define a bilinear form $B[t;\cdot,\cdot] : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ by

$$B[t;u,v] = \int_\Omega \left\{ a^{i,j}(x,t)u_{x_i}v_{x_j} + b^i(x,t)u_{x_i}v + c(x,t)uv \right\} dx$$

Likewise, we may regard a solution $u = u(x,t)$ as a mapping $t \mapsto \mathbf{u}(t) := u(\cdot,t)$ from $[0,T]$ into the function space $\mathbf{X} = H_0^1(\Omega)$. Thus a solution can be regarded as a trajectory in $\mathbf{X}$. Indeed, when L is in the divergence form, it is very convenient to work on the space $u \in L^2(0,T;H_0^1(\Omega))$. Similarly, we regard $\mathbf{f}(t) = f(\cdot,t)$ as a function in $L^2(0,T;H^{-1}(\Omega))$.

3.1.1 Definiton of Weak solutions

Definition 3.1. *Assume that $g \in L^2(\Omega)$ and $\mathbf{f} \in L^2(0,T;H^{-1}(\Omega))$. We say that a function*

$$\mathbf{u} \in L^2(0,T;H_0^1(\Omega)) \quad \textit{with} \quad \mathbf{u}' \in L^2(0,T;H^{-1}(\Omega)),$$

is a weak solution of the initial boundary value problem (3.6) if $\mathbf{u}(0) = g$ in $L^2(\Omega)$ and

$$\langle \mathbf{u}', v\rangle + B[t;\mathbf{u},v] = (\mathbf{f},v)$$

for each $v \in H_0^1(\Omega)$ and a.e. $0 \le t \le T$.

Here the requirement $\mathbf{u}(0) = g$ makes sense since by Theorem 1.18, $\mathbf{u} \in C([0,T];L^2(\Omega))$.

3.1.2 Existence of Weak Solutions

The method we employ for proving existence is known as the Galerkin method[1] which approximates the Banach $\mathbf{X} = H_0^1(\Omega)$ by finite space dimensional subspace $\mathbf{X}_m$ and project the evolution equation over $L^2(0,T;\mathbf{X}_m)$

[1]Another useful method is to discritize the time by either an Euler forward or an Euler backward scheme, and solving the resulting elliptic equations. In the current situation, it seems that the Galerkin method offers some advantages, although in general both methods seem very robust.

Let $\{w_k\}_{k=1}^{\infty}$ be an orthonormal basis for $L^2(\Omega)$ which is also an orthogonal basis for $H_0^1(\Omega)$. Consider

$$\mathbf{u}_m(t) := \sum_{k=1}^{m} d_m^k(t) w_k,$$

where the $d_m^k(t)$ are real-valued functions of t. We shall choose coefficients $d_m^k(t)$ in such a way that $\mathbf{u}_m(t)$ is an approximate solution of our boundary value problem.

Specifically, for each fixed $m \geq 1$, we seek $d_m^k(t), k = 1, \cdots, m$ such that for all $k = 1, \cdots, m$,

$$(\mathbf{u}_m', w_k) + B[t; \mathbf{u}_m, w_k] = \langle \mathbf{f}, w_k \rangle \tag{3.8}$$

$$d_m^k(0) = (g, w_k)_{L^2}. \tag{3.9}$$

Lemma 3.1. *For each m, there is a unique function $\mathbf{u}_m$ of the prescribed form satisfying (3.8) and (3.9) for all $k = 1, \cdots, m$.*

Proof. We have

$$(\mathbf{u}_m'(t), w_k) = {d_m^k}'(t),$$

and

$$B[t; \mathbf{u}_m, w_k] = \sum_{l=1}^{m} e^{kl}(t) d_m^l(t)$$

for $e^{kl}(t) = B[t; w_l, w_k]$. Set $f^k(t) = (\mathbf{f}(t), w_k)$. The system becomes

$${d_m^k}'(t) + \sum_{l=1}^{m} e^{kl}(t) d_m^l(t) = f^k(t)$$

with the appropriate initial conditions. By the existence and uniqueness theorem from ODEs, there exists an absolutely continuous function $\mathbf{d}_m(t) = (d_m^1(t), ..., d_m^m(t))$ which solves this system. □

We now show that as $m \to \infty$, $\mathbf{u}_m$ converges to a weak solution of (3.6). To do this, we assume that $\mathcal{L}$ is elliptic in the sense that there exists positive constants α, β and μ such that for (a.e.) $t \in [0, T]$

$$B[t; u, v] \leq \alpha \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)} \quad \forall\, u, v \in H_0^1(\Omega),$$

$$B[t; u, u] \geq \beta \|u\|^2_{H_0^1(\Omega)} - \mu \|u\|^2_{L^2(\Omega)} \quad \forall\, u \in H_0^1(\Omega).$$

We make use of the following estimate.

Lemma 3.2 (Energy estimate). *There exists $C = C(\Omega, T, L)$ such that for each integer $m \geq 1$,*

$$\max_{0 \leq t \leq T} \|\mathbf{u}_m(t)\|_{L^2(\Omega)} + \|\mathbf{u}_m\|_{L^2(0,T;H_0^1(\Omega))} + \left\|\mathbf{u}'_m\right\|_{L^2(0,T;H^{-1}(\Omega))}$$
$$\leq C(\|\mathbf{f}\|_{L^2(0,T;H^{-1}(\Omega))} + \|g\|_{L^2(\Omega)})$$

Proof sketch. Multiplying (3.8) by d_m^k and sum over $k = 1, \cdots, m$ one obtains

$$(u_m, u_m') + B[t; u_m, u_m] = \langle \mathbf{f}, u_m \rangle.$$

First of all, we have

$$(\mathbf{u}'_m, \mathbf{u}_m) = \frac{d}{dt}\left(\frac{1}{2}\|\mathbf{u}_m\|^2_{L^2(\Omega)}\right).$$

Secondly, by the ellipticity of $\mathcal{L}$,

$$B[t; \mathbf{u}_m, \mathbf{u}_m] \leq \beta \|u_m\|^2_{H_0^1(\Omega)} - \gamma \|\mathbf{u}_m\|^2_{L^2(\Omega)}$$

Finally,

$$|\langle \mathbf{f}, \mathbf{u}_m \rangle| \leq \|\mathbf{f}\|_{H^{-1}(\Omega)} \|u_m\|_{H_0^1(\Omega)} \leq \tfrac{\beta}{2} \|\mathbf{u}_m\|^2_{H_0^1(\Omega)} + \frac{1}{2\beta}\|\mathbf{f}\|^2_{H^{-1}(\Omega)}.$$

Using these inequalities, we readily obtain an inequality of the form

$$\eta'(t) \leq C_1 \eta(t) + C_2 \xi(t)$$

where

$$\eta(t) = \|\mathbf{u}_m\|^2_{L^2(\Omega)}, \qquad \xi(t) = \|\mathbf{f}\|^2_{L^2(\Omega)},$$

whence by Gronwall's inequality,

$$\eta(t) \leq e^{C_1 t}\left(\eta(0) + C_2 \int_0^t \xi(s) ds\right).$$

Noticing that $\eta(0) = \|\mathbf{u}_m(0)\|_{L^2(\Omega)} \leq \|g\|_{L^2(\Omega)}$ implies

$$\max_{0 \leq t \leq T} \|\mathbf{u}_m(t)\|_{L^2(\Omega)} \leq C(\|\mathbf{f}\|_{L^2(0,T;L^2(\Omega))} + ||g||_{L^2(\Omega)}).$$

Estimates for the other terms follow via elementary calculations. □

Theorem 3.1. *There exists a weak solution of the initial boundary value problem (3.6).*

Proof sketch. Using the Galerkin approximation, one obtains a sequence $\{\mathbf{u}_m\}_{m=1}^{\infty}$ of approximation solutions. By the energy estimates, this sequence of approximate solutions is bounded in $L^2(0, T; H_0^1(\Omega))$, and therefore weakly precompact, so there exists a weakly convergent subsequence $\mathbf{u}_{m_l}$ such that

$$\begin{aligned} \mathbf{u}_{m_l} &\xrightarrow{w} \mathbf{u} && \text{weakly in } L^2(0, T; H_0^1(\Omega)) \\ \mathbf{u}'_{m_l} &\xrightarrow{w} \mathbf{u}' && \text{weakly in } L^2(0, T; H^{-1}(\Omega)) \end{aligned}$$

Now one uses the density of functions

$$\sum_1^N d^k(t) w_k$$

in the space $L^2(0, T; H_0^1(\Omega))$ to prove that the weak limit is a weak solution of the problem. □

3.1.3 Uniqueness

Theorem 3.2. *A weak solution of the problem is unique.*

Proof sketch. Assume that we have a weak solution $\mathbf{u}$ with

$$\mathbf{f} = g = 0.$$

We have

$$\frac{d}{dt}\left(\frac{1}{2}\|\mathbf{u}\|^2_{L^2(\Omega)}\right) + B[t;\mathbf{u},\mathbf{u}] = \langle \mathbf{u}', \mathbf{u}\rangle + B[t;\mathbf{u},\mathbf{u}] = 0$$

and also

$$B[t;\mathbf{u},\mathbf{u}] \geq \beta\|\mathbf{u}\|^2_{H^1_0(\Omega)} - \gamma\|\mathbf{u}\|^2_{L^2(\Omega)} \geq -\gamma\|\mathbf{u}\|^2_{L^2(\Omega)}$$

whence the result follows from Gronwall's inequality. □

3.1.4 Regularity in Hilbert Spaces

By regarding $-(b^j u)_{x_i} + b^i u_{x_i} + cu$ as source terms, we assume that

$$Lu = -(a^{ij}u_{x_i})_{x_j}.$$

We assume that L is uniformly elliptic and for m under consideration, $D^\alpha_x D^l_t a^{ij} \in C^0(\bar{\Omega}_T)$ for all $|\alpha| + 2l \leq 2m+2$. Also, we assume that $\partial\Omega \in C^{m+2}$.

Theorem 3.3. *Let m be a non–negative integer. Assume*

$$g \in H^{2m+1}(\Omega),\ \frac{d^k\mathbf{f}}{dt^k} \in L^2(0,T;H^{2m-2k}(\Omega)) \quad \forall\, k = 0,\cdots,m.$$

Suppose also we have the compatibility conditions

$$\begin{cases} g_0 := g \in H^1_0(\Omega),\ g_1 := \mathbf{f}(0) - Lg_0 \in H^1_0(\Omega), \\ \ldots, g_m := \dfrac{d^{m-1}\mathbf{f}}{dt^{m-1}}(0) - Lg_{m-1} \in H^1_0(\Omega) \end{cases}$$

Then

$$\frac{d^k\mathbf{u}}{dt^k} \in L^2(0,T;H^{2m+2-2k}(\Omega)) \quad k = 0,1,...,m-1$$

and we have the estimate

$$\sum_{k=0}^{m+1}\left\|\frac{d^k\mathbf{u}}{dt^k}\right\|_{L^2(0,T;H^{2m+2-2k}(\Omega))} \leq C\left(\sum_{k=0}^{m}\left\|\frac{d^k\mathbf{f}}{dt^k}\right\|_{L^2(0,T;H^{2m-2k}(\Omega))} + ||g||_{H^{2m+1}(\Omega)}\right)$$

Using Sobolev embeddings, this theorem allows us to prove smoothness of solutions (for instance) under the assumption that the data of the problem are smooth.

3.2 Parabolic Equation in Non-divergence form

Not discussed in class.

3.3 Parabolic A Priori Estimates

We include here the precise statements (without proof) of some of the regularity theorems for parabolic PDEs, although such theorems had been only suggested in the lectures.

In each of the estimates given below, there are two versions, one is called local, which does not need initial boundary values, and the other called global, which need initial and boundary values. We state the global version only.

3.3.1 The Hölder Estimate

Theorem 3.4. *Assume that $\mathcal{L}$ is uniformly elliptic in divergence or non-divergence form. Then the solution is Hölder continuous.*

3.3.2 The L^p Estimate

Theorem 3.5 (L^p estimate). *Assume that*

$$\mathcal{L}u = -a^{ij}u_{x_i x_j} + b^i u_{x_i} + cu$$

where $\mathcal{L}$ is uniformly elliptic and

$$a^{ij} \in C^0(\bar{\Omega}_T), \quad b^i \in L^\infty, \quad c \in L^\infty, \quad f \in L^p, \quad g \in W_p^{2,1},$$

and that $\partial\Omega \in C^2$. Also asume that $g \in W^{2,p} \cap H_0^1$. Then we have the estimate

$$\|u\|_{W_p^{2,1}} := \sum_{|\alpha|+2l \leq 2} \|D_x^\alpha D_t^l u\|_{L^p} \leq C\{\|f\|_{L^p} + \|g\|_{W^{2,p}}\}.$$

3.3.3 The Schauder Estimate

Theorem 3.6 (Schauder estimate). *Let $\alpha \in (0,1)$. Set $Q = \Omega \times (0,T]$ Assume that $a^{ij}, b^i, c \in C^{\alpha,\alpha/2}(\bar{Q})$ and that $f \in C^{\alpha,\alpha/2}(\bar{Q})$, $g \in C^{2+\alpha}(\bar{\Omega}) \cap H_0^1(\Omega)$, $\partial\Omega \in C^{2+\alpha}$, then we $u \in C^{2+\alpha,1+\alpha/2}(\bar{Q})$ and*

$$\|u\|_{C^{2+\alpha,1+\alpha/2}(\bar{Q})} \leq C[\ \left\{\|f\|_{C^{\alpha,\alpha/2}(\bar{Q})} + \|g\|_{C^{2+\alpha}(\bar{\Omega})}\right\}.$$

3.4 Hyperbolic Equations

As before, let $\Omega_T = \Omega \times (0,T]$. We shall be interested in the initial boundary-value problem

$$\begin{cases} u_{tt} + Lu = f & \text{in } \Omega_T \\ u = 0 & \text{in } \partial\Omega \times [0,T] \\ u = g,\ u_t = h & \text{in } \Omega \times \{t = 0\} \end{cases}$$

For simplicity, we shall assume that L is a second order uniformly elliptic operator. In this case, we say that the problem is (uniformly) hyperbolic.

3.4.1 Definition of Weak Solutions

For simplicity, we assume that $\mathcal{L}$ is in divergence form.

As in the case of parabolic equations, we define weak solutions in terms of curves in Banach spaces. Specifically, let

$$\mathbf{u} : [0,T] \to H_0^1(\Omega)$$

be defined by

$$\mathbf{u}(t) = u(.,t) \in H_0^1(\Omega),$$

and

$$\mathbf{f} : [0,T] \to L^2(\Omega)$$

by

$$\mathbf{f}(t) = f(.,t) \in L^2(\Omega).$$

Definition 3.2. *A function*

$$\mathbf{u} \in L^2(0,T; H_0^1(\Omega))$$

with

$$\mathbf{u}' \in L^2(0,T; L^2(\Omega)), \quad \mathbf{u}'' \in L^2(0,T; H^{-1}(\Omega))$$

is said to be a weak solution of the hyperbolic problem if

$$(\mathbf{u}'', v) + B[t; \mathbf{u}, v] = \langle \mathbf{f}, v\rangle \quad \forall v \in H_0^1(\Omega)$$

for (a.e.) $t \in [0,T]$, and

$$\mathbf{u}(0) = \mathbf{g}, \quad \mathbf{u}'(0) = \mathbf{h}.$$

Note that in light of an earlier theorem on curves in Banach spaces, the latter two conditions make sense, becase $\mathbf{u}$ and $\mathbf{u}'$ are *continuous.*

3.4.2 Existence Weak Solutions

Again we can employ the Galerkin approximation. Thus, we attempt to construct a sequence $\mathbf{u}_m$ of approximate solutions of the form

$$\mathbf{u}_m(t) = \sum_{k=1}^{m} d_m^k(t) w_k$$

where w_k is an orthogonal basis for $H_0^1(\Omega)$, and an orthonormal basis of $L^2(\Omega)$. The coefficients are real-valued functions which are required to obey the initial conditions

$$d_m^k(0) = (g, w_k)$$

$$d_m^k{}'(0) = (h, w_k),$$

and the $\mathbf{u}_m$ are to obey the differential equation

$$(\mathbf{u}_m'', w_k) + B[\mathbf{u}_m, w_k; t] = (\mathbf{f}, w_k).$$

By using the fundamental existence and uniqueness theorem from ODEs, this system of initial-value problems is uniquely solvable for the coefficients $d_m^k(t)$.

Moreover, as before we obtain the energy estimate which is key to proving that the sequence $\mathbf{u}_m$ has a weak limit point which is a weak solution of the desired problem:

Lemma 3.3. *(Energy estimate) We have*

$$\max_{0\le t\le T}\Big\{\|\mathbf{u}_m(t)\|_{H_0^1(\Omega)}+\left\|\mathbf{u}'_m(t)\right\|_{L^2(\Omega)} Big\}$$
$$+\left\|\mathbf{u}''_m\right\|_{L^2(0,T;H_0^1(\Omega))}+\left\|\mathbf{u}'_m\right\|_{L^2(0,T;H^{-1}(\Omega))}$$
$$\le C(\|\mathbf{f}\|_{L^2(0,T;L^2(\Omega))}+\|g\|_{H_0^1(\Omega)}+\|h\|_{L^2(\Omega)})$$

Idea of Proof. The idea of the proof is similar to the parabolic version: obtain estimates which enable one to apply Gronwall's inequality. □

Using the energy estimates, we may prove that u_m converges to a weak solution. Hence, we have the following existence result.

Theorem 3.7. *A weak solution of the hyperbolic problem exists.*

The proof is completely analogous to the proof for the parabolic case.

3.4.3 Uniqueness

Theorem 3.8. *Solutions are unique.*

Again, the proof relies upon deriving an estimate in the form of Gronwall's inequality. The details are somewhat tricky, and we leave them out.

3.4.4 Regularity

3.5 Introduction to Semigroup Theory

In studying evolution equations, a space time function $u(x,t)$ can be simply viewed as a funtion from t to $\mathbf{u}(t) := u(\cdot,t)$ in certain Sobolev space $\mathbf{X}$, say $H^1(\Omega)$. That is, a linear evolution equation can be written as

$$\mathbf{u}_t = A\mathbf{u} \tag{3.10}$$

where A is a linear operator. Hence, one can pretty much regard the evolution equation as an ode and the solution $\mathbf{u}$ as a curve (trajectory) in certain Banach space. Indeed, a lot of the ode ideas can be used.

In this chapter, we introduce one of the very powerful tools in studying evolution equations, the semigroup theory.

3.5.1 Semigroups and Their Generators

Definition 3.3. *Let* $\mathbf{X}$ *be a Banach space. A familly* $\{S(t)\}_{t\geq 0}$ *of bounded linear operators from* $\mathbf{X}$ *to* $\mathbf{X}$ *is called a semigroup if the following holds:*

1. $S(0)u = u$ *for every* $u \in \mathbf{X}$*;*

2. $S(t+s)u = S(t)S(s)u = S(s)S(t)u$ *for every* $u \in \mathbf{X}$ *and* $t, s \geq 0$

3. *For each* $u \in \mathbf{X}$*, the maping* $t \to S(t)u$ *is continuous from* $[0, \infty)$ *into* $\mathbf{X}$*.*

Definition 3.4. *A semigroup* $\{S(t)\}_{t\geq 0}$ *is called a contraction semigroup if*

$$\|S(t)\| \leq 1 \quad \forall\, t \geq 0 \tag{3.11}$$

where $\|\cdot\|$ *stands for the operatorial norm from* $\mathbf{X}$ *to* $\mathbf{X}$*.*

Definition 3.5. *Let* $\{S(t)\}_{t\geq 0}$ *be a semigroup on a Banach space* $\mathbf{X}$*. Define*

$$D(A) := \left\{ u \in \mathbf{X}\,;\ \lim_{t\to 0+} \frac{S(t)u - u}{t} \text{ exists in } \mathbf{X} \right\} \tag{3.12}$$

and $A : D(A) \to \mathbf{X}$ *by*

$$Au := \lim_{t\to 0+} \frac{S(t)u - u}{t} \quad \forall u \in D(A). \tag{3.13}$$

This linear operator $A : D(A) \to \mathbf{X}$ *is called the (infinitesimal) generator of the semigroup* $\{S(t)\}_{t\geq 0}$ *and* $D(A)$ *the definition domain of* A*.*

Example 3.1. Let $\lambda \in \mathbb{R}$ (or $\mathbb{C}$) be a number. Define $S(t)u = e^{\lambda t}u$ for all $t \in \mathbb{R}$ and $u \in \mathbf{X}$. Then $\{S(t)\}_{t\geq 0}$ is a semigroup and its generator is $\lambda\mathbf{I}$ where $\mathbf{I}$ is the identity operator. The definition domain of the generator is the whole space $\mathbf{X}$. The semigroup is a contraction if and only if $\Re(\lambda) \leq 0$.

Example 3.2. Let A be a bounded operator from $\mathbf{X}$ to $\mathbf{X}$. Define, for any $t \in \mathbb{R}$ and $u \in \mathbf{X}$,

$$S(t)u = \sum_{k\geq 0} \frac{t^k A^k}{k!} u.$$

Then one can verify that $\{S(t)\}_{t\geq 0}$ is a semigroup with generator A whose definition domain is $\mathbf{X}$. Typically, we write the operator $S(t)$ as e^{At}.

Now if $(\lambda, \phi) \in \mathbb{C} \times \mathbf{X}$ is an eigen pair of A, i.e., $A\phi = \lambda\phi$, then $e^{At}\phi = e^{\lambda t}\phi$.

Notice that for every $u_0 \in \mathbf{X}$, the function $u(t) := S(t)u_0$ solves the initial value problem

$$u_t = Au \quad \forall\, t > 0, \quad u(0) = u_0.$$

We remark that in these two examples, $\{S(t)\}_{t\in\mathbb{R}}$ is indeed a group.

Example 3.3. Let $\Omega = (0, \pi)$ and $\mathbf{X} = L^2(\Omega)$. For each integer $n \geq 1$ define $\phi_n = \sqrt{\frac{2}{\pi}} \sin(nx)$ and $\psi_n = \sqrt{\frac{2}{\pi}} \cos(nx)$ and $\psi_0 = \sqrt{\frac{1}{\pi}}$. Then both $\{\phi_n\}_{n=1}^{\infty}$ and $\{\psi_n\}_{n=0}^{\infty}$ are orthonomal bases of $\mathbf{X}$. Let $A = \frac{d^2}{dx^2}$. Then we see that $A\phi_n = \lambda_n \phi_n$ and $A\psi_n = \lambda_n \psi_n$ where $\lambda_n = -n^2$.

For each $t \geq 0$ and $u \in \mathbf{X}$, we define

$$\begin{aligned} S(t)u &= \sum_{n\geq 1} e^{\lambda_n t}(u, \phi_n)\phi_n, \\ \tilde{S}(t)u &= \sum_{n\geq 0} e^{\lambda_n t}(u, \psi_n)\psi_n. \end{aligned}$$

Then one can verify that both $\{S(t)\}_{t\geq 0}$ and $\{\tilde{S}(t)\}_{t\geq 0}$ are contraction semigroups, and both have A as their generators. Nevertheless, the definition domains of the generators for the two semigroups are different. For $\{S(t)\}_{t\geq 0}$, the definition domain of the generator is $H^2(\Omega) \cap H_0^1(\Omega)$, where as for $\{\tilde{S}(t)\}_{t\geq 0}$, it is $\{u \in H^2(\Omega); u'(0) = u'(\pi) = 0\}$.

One can check that for any $u_0 \in \mathbf{X}$, $u(t) := S(t)u_0$ and $\tilde{u}(t) := \tilde{S}(t)u_0$ for all $t \geq 0$ solve, respectively

$$\begin{cases} u_t = u_{xx} & \text{in } (0, \pi) \times (0, \infty), \\ u(0, t) = u(\pi, t) = 0 & \text{on } \{0, \pi\} \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{on } [0, \pi] \times \{0\}, \end{cases}$$

$$\begin{cases} \tilde{u}_t = \tilde{u}_{xx} & \text{in } (0, \pi) \times (0, \infty), \\ \tilde{u}_x(0, t) = \tilde{u}_x(\pi, t) = 0 & \text{on } \{0, \pi\} \times (0, \infty), \\ \tilde{u}(x, 0) = u_0(x) & \text{on } [0, \pi] \times \{0\}. \end{cases}$$

From this example, one sees that the specification of the definition domain of A is very important.

Homework 3.1. *Verify the conclusions stated in example 3.3.*

3.5.2 Properties of Generators

Theorem 3.9. *Let A be the generator of a semigroup $\{S(t)\}_{t\geq 0}$ on a banach space $\mathbf{X}$. Then, for each $u \in D(A)$, the following holds:*

1. *For each $t \geq 0$, $S(t)u \in D(A)$ and $AS(t)u = S(t)Au$.*

2. *Define $x(t) = S(t)u$ for all $t \geq 0$. Then $x \in C^1[0, \infty; \mathbf{X})$ and*

$$\dot{x}(t) = A\, x(t) \quad \forall\, t \geq 0, \qquad x(0) = u.$$

Theorem 3.10. *Assume that A is the generator of a contraction semigroup on $\mathbf{X}$. Then*

1. *$D(A)$ is dense in $\mathbf{X}$, and*

2. *A is a closed operator.*

Definition 3.6. *Let $\mathbf{X}$ be a banach space and $A : D(A) \subset \mathbf{X} \to \mathbf{X}$ be linear.*

1. *A real number* λ *belongs to* $\rho(A)$, *the resolvent set of* A, *if*

$$\lambda I - A : D(A) \to X \quad \text{is one to one and onto.}$$

2. *If* $\lambda \in \rho(A)$, *the resolvent operator* $R_\lambda : \mathbf{X} \to D(A)$ *is defined by*

$$R_\lambda u := (\lambda I - A)^{-1} u \qquad \forall u \in \mathbf{X}.$$

Now suppose that A is a closed operator and λ is in the resolvant set of A. Then it is an easy consequence of the Closed Mapping Theorem that R_λ is a bounded linear operator from $\mathbf{X} \to D(A) \subset \mathbf{X}$. Moreover, A and R_λ commute i.e.

$$AR_\lambda u = R_\lambda A u \qquad \forall u \in D(A).$$

Theorem 3.11. *Let* A *be a generator of a contraction semigroup* $\{S(t)\}_{t\geq 0}$.

1. *If* $\lambda, \mu \in \rho(A)$ *then*
$$R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu \tag{3.14}$$
and
$$R_\lambda R_\mu = R_\mu R_\lambda \tag{3.15}$$

2. *If* $\lambda > 0$ *then* $\lambda \in \rho(A)$ *and*
$$R_\lambda u = \int_0^\infty e^{-\lambda t} S(t) u dt \quad \forall u \in \mathbf{X}. \tag{3.16}$$
Consequentrly, $\|R_\lambda\| \leq \lambda^{-1}$.

3.5.3 The Hille–Yosida Theorem

Theorem 3.12. (Hille Yosida Theorem) *Let* A *be a closed ,densely–defined linear operator on a Banach space* $\mathbf{X}$. *Then* A *is the generator of a contraction semigroup* $\{S(t)\}_{t\geq 0}$ *if and only if*

$$(0,\infty) \subset \rho(A) \quad \text{and} \quad \|R_\lambda\| \leq \lambda^{-1} \quad \forall\, \lambda > 0. \tag{3.17}$$

Definition 3.7. *A semigroup* $S(t)_{t\geq 0}$ *is called* ω*-contractive if*

$$\|S(t)\| \leq e^{\omega t} \quad \forall\, t \geq 0. \tag{3.18}$$

The necessary and sufficient condition for a closed, densely defined operator A to generate an ω-contractive semigroup is

$$(\omega, \infty) \subset \rho(A) \quad \text{and} \quad \|R^\lambda(A)\| \leq \frac{1}{\lambda - \omega} \;\; \forall \lambda > \omega. \tag{3.19}$$

3.5.4 Applications

Example 3.4. Consider the heat equation

$$\begin{cases} u_t = \Delta u & \text{on } \Omega \times (0, \infty) \\ u = 0 & \text{on } \partial\Omega \times [0, \infty) \\ u(x, 0) = u_0(x) & \text{on } \Omega \times \{0\}. \end{cases} \tag{3.20}$$

Set

$$\mathbf{X} = L^2(\Omega), \quad D(A) = H^2(\Omega) \cap H_0^1(\Omega), \quad Au = \Delta u.$$

Then A is closed and $D(A)$ is dense in $\mathbf{X}$. Also, for every $\lambda > 0$, $(\lambda I - A)^{-1}$ from $\mathbf{X}$ to $D(A)$ exists.

To estimate $\|R_\lambda\|$ consider the equation

$$\lambda u - \triangle u = f \in L^2(\Omega), \quad u \in H_0^1(\Omega).$$

We multiply by u and we integrate by parts to get

$$\lambda \|u\|^2 + \|\nabla u\|^2 = (f, u) \leq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}. \tag{3.21}$$

Hence

$$\|u\|_{L^2} \leq \frac{1}{\lambda} \|f\|_{L^2}$$

which implies $\|R_\lambda\| \leq \frac{1}{\lambda}$.

Consequently the Hille–Yosida theorem implies that A generates a semigroup $\{S(t)\}_{t \geq 0}$ on $\mathbf{X}$. Hence, for every $u_0 \in L^2(\Omega)$, $u(t) := S(t)u_0$ for all $t \geq 0$ is a solution to our problem.

Example 3.5. Consider the following parabolic equation in non–divergence form

$$\begin{cases} u_t = -\mathcal{L}u := a^{ij} u_{x_i x_j} - b^i u_{x_i} - cu & \text{in } \Omega \times (0, \infty), \\ u = 0 & \text{on } \partial\Omega \times [0, \infty) \\ u(x, 0) = u_0(x) & \text{on } \Omega \times \{0\} \end{cases} \tag{3.22}$$

where $a^{ij} \in C^0(\bar{\Omega})$ and $b^i, c \in L^\infty(\Omega)$ are independent of t, and for some positive constant θ, $a^{ij}(x)\xi_i \xi_j \geq \theta |\xi|^2$, for every $\xi \in \mathbb{R}^n$ and $x \in \bar{\Omega}$.

Set $\mathbf{X} = L^2(\Omega)$, $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$ and $A = -\mathcal{L}$, i.e.,

$$A : u \in D(A) \to A(u) = a^{ij} u_{x_i x_j} - b^i u_{x_i} + cu \in \mathbf{X}. \tag{3.23}$$

It is easy to see that $D(A)$ is dense in $\mathbf{X}$ and A is a closed operator. Also, by ellptic theory, there exists $\omega \in \mathbb{R}$ such that

$$(\omega, \infty) \subset \rho(A) \quad \text{and} \quad \|R^\lambda(A)\| \leq \frac{1}{\lambda - \omega} \quad \forall \lambda > \omega. \tag{3.24}$$

Hence, the Hille–Yosida theorem can be applied to give solutions to the intial value problem for each $u_0 \in L^2(\Omega)$.

Example 3.6. Consider the wave equation

$$\begin{cases} u_{tt} = \triangle u & \text{on } \Omega \times (0,\infty) \\ u = 0 & \text{on } \partial\Omega \times [0,\infty) \\ u(x,0) = u_0(x) & \text{on } \Omega \times \{0\} \\ u_t(x,0) = v_0(x) & \text{on } \Omega \times \{0\} \end{cases} \tag{3.25}$$

To use the Hille–Yosida theorem, we first transfer the second order (in time) scalar equation in to a first order (in time) system:

Set $v = u_t$ and $U = (u, v)$. Then

$$U_t = (u,v)_t = (v, \triangle u) =: AU.$$

Also $U_0 = (u_0, v_0)$.

Set $\mathbf{X} = H_0^1(\Omega) \times L^2(\Omega)$, and $D(A) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$.

It can be proven that A satisfies the conditions of Hille-Yosida theorem and consequently A generates a semigroup on $\mathbf{X} = H_0^1(\Omega) \times L^2(\Omega)$; namely, the wave equation is solvable for every initial data $(u_0, v_0) \in \mathbf{X}$.

Chapter 4

SOME TECHNIQUES FOR NONLINEAR PDES

In studyiny non–linear PDEs, quite often one uses theories of linear PDEs to transfer the existence problem into a fixed point of some map. In this chapter, we introduce in the first section some very commonly used fixed point theorems— the contraction mapping, the Schauder, and the Lera–Schauder. In subsequent sections, we provides several non–linear PDE examples demonstrating their applications. In these applications, various kinds of other commonly used techniques are also demonstrated.

4.1 Fixed Point Theorems

4.1.1 Contraction Mapping

Theorem 4.1. *Assume that B is a closed subset of a Banach space $\mathbf{X}$ and $\mathbf{T} : B \to B$ is a contraction mapping, i.e., there exists $\theta \in (0, 1)$ such that*

$$\|\mathbf{T}x - \mathbf{T}y\| \le \theta\|x - y\| \quad \forall x, y \in B.$$

Then there is a unique $x \in B$ such that $x = \mathbf{T}x$.

4.1.2 The Schauder's Fixed Point Theorem

We recall that a set is *precompact* if every sequence in the set contains a convergent subsequence. A set is *compact* if it is precompact and closed.

Theorem 4.2. *Let $\mathbf{X}$ be a Banach space, B be a closed and convex subset of $\mathbf{X}$, and $\mathbf{T} : B \mapsto B$ be a continuous map. Then $\mathbf{T}$ has at least one fixed point in B if the range $\mathbf{T}(B) := \{Tx; x \in B\}$ is precompact.*

Note that if B is compact, then automatically $\mathbf{T}(B) \subset B$ is precompact.

In both the contraction and the Schauder fixed point theorems, the condition that $\mathbf{T}$ maps B into itself is usually the key difficulties in applications.

The central point of the Schauder's fixed point theorem is the compactness of the image $\mathbf{T}B$. Quite often one takes a bounded B and shows that $\mathbf{T}$ is compact. In other cases, one simply takes a compact B (then one has to be very careful about the continuity of $\mathbf{T}$).

4.1.3 The Leray–Schauder Fixed Point Theorem

Theorem 4.3. *Let $\mathbf{X}$ be a Banach space and $\mathbf{T} : \mathbf{X} \to \mathbf{X}$ be compact and continuous. Suppose there exists a positive constant M such that*

$$\|x\| \leq M \text{ whenever } \exists\, \sigma \in [0,1] \text{ such that } x = \sigma T x.$$

Then $\mathbf{T}$ has a fixed point in $\mathbf{X}$.

The Leray–Schauder's fixed point theorem is also known as the Schaefer's fixed point theorem. Its advantage over the Schauder's fixed point theorem for applications is that we do not have to identify a convex closed subset which T maps into itself.

4.1.4 Examples

Here we provide two simple examples to demonstrate the applications of the three fixed point theorems.

Example 4.1. Consider the initial value problem for an ODE system, for $\mathbf{x} : t \mapsto \mathbf{x}(t) \in \mathbb{R}^n$,

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, t), & t \in \mathbb{R}, \\ \mathbf{x}(t_0) = x_0 \end{cases} \tag{4.1}$$

where $\mathbf{f} : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is continuous. It is easy to verify that (4.1) is equivalent to find $\mathbf{x}$ such that

$$\mathbf{x}(t) = \mathbf{T}\mathbf{x} := x_0 + \int_{t_0}^{t} \mathbf{f}(\mathbf{x}(\tau), \tau)\, d\tau. \tag{4.2}$$

For positive constants δ and ϵ to be determined, one defines

$$\mathbf{X} = C([t_0 - \delta, t_0 + \delta]), \qquad B = \{\mathbf{x} \in \mathbf{X};\, \|\mathbf{x} - x_0\|_{C([t_0-\delta,t_0+\delta])} \leq \epsilon\}.$$

Then for approriate small δ and ϵ, one can verify that $\mathbf{T}$ is continuous and maps B into itself.

We consider separtely the following senearios:

A) Suppose that $D_x\mathbf{f}$ is continuous (or a little weaker, $\mathbf{f}$ is Lipschitz in x with uniform Lipschitz constant in every bounded subset of $\mathbb{R}^{n+1}$). Then one can show that $\mathbf{T}$ is a contraction (for approriate δ and ϵ), and therefore the contraction mapping theorem for $\mathbf{T}$ gives the local in time unique existence of solutions to the initial value problem.

B) Suppose that we drop the Lipschitz (in x) continuous condition of f. Then we cannot apply the contraction mapping theorem. But since the image $\mathbf{T}B$ is a bounded set in $C^1([t_0 - \delta, t_0+\delta])$, we can apply the Schauder's fixed point theorem to establish the local in time existence of the solution.

In this example, we can explicitly construct the set B such that $\mathbf{T}$ maps B into itself.

Example 4.2. Consider the boundary value problem, for $u = u(x)$,

$$\begin{cases} u'' = f(u,x), & x \in (0,L), \\ u(0) = u(L) = 0 \end{cases} \tag{4.3}$$

where $f \in C(\mathbb{R} \times [0,L])$.

Let $\mathbf{X} = C([0,L])$ and for each $v \in \mathbf{X}$, we let $u = \mathbf{T}v$ to be the solution to

$$u'' = f(v,x) \ \text{ in } \ (0,L), \qquad u(0) = u(L) = 0.$$

Namely,

$$\mathbf{T}v = \frac{(x-L)}{L}\int_0^x \xi f(v(\xi),\xi)\,d\xi + \frac{x}{L}\int_L^x (L-\xi) f(v(\xi),\xi)\,d\xi, \quad x \in [0,L].$$

Then $\mathbf{T}$ is continuous and compact.

A) Suppose that $f(u,x)$ does not depend on u very strongly; more precisely, there exists $\theta \in (0, 8/L^2)$ such that $\sup|f_u| \le \theta$. Then it is easy to verify that $\mathbf{T}$ is a contraction and hence taking $B = \mathbf{X}$ we obtain the existence of a unique solution, which is the unique fixed point of $\mathbf{T}$.

B) Suppose we only know that f_u is continuous. Then for sufficiently small L, one can construct a bounded closed convex set $B \subset \mathbf{X}$ such that $\mathbf{T}$ maps B into itself and is a contraction (in B). Hence, the contraction mapping theorem can be used to show the existence and local uniqueness. Nevertheless, when L is large, the contraction mapping theorem does not apply.

C) Suppose f is continuous and uniformly bounded. Then it is easy to see that $\mathbf{TX}$ is precompact in $\mathbf{X}$. Hence, taking any convex closed B such that $\mathbf{X} \subset B$, we can use the Schauder theorem to show the existence of at elast a solution.

D) Suppose that f is continuous and

$$u\, f(u,x) > 0 \qquad \forall\, x \in [0,L], u \in \mathbb{R} \setminus [-1,1].$$

Then we can use the Laray–Schauder fixed point theorem. Indeed, suppose for some $\sigma \in [0,1]$ and $v \in \mathbf{X}$ we have $v = \sigma \mathbf{T} v$. Then $v \in C^2([0,L])$ and $v'' = \sigma f(v,x)$ in $(0,L)$ and $v(0) = v(L) = 0$. By the maximum principle, we find that $\max_{x\in[0,L]} v(x) \le 1$ and $\min_{x\in[0,L]} v(x) \ge -1$ since $f(v,x) > 0$ whenever $v > 1$ and $f(v,x) < 0$ whenever $v < -1$. Thus, $\|v\|_{\mathbf{X}} \le M := 1$. The Leray–Schauder's fixed point theorem then implies that $\mathbf{T}$ has at least a fixed point, which yields a solution to the boundary value problem.

Note that in this case, the Schauder's fixed point theorem can hardly be applied. The contraction mapping cannot be used since we did not assume that f is Lipschitz in u.

Homework 4.1. *Suppose that f is continuous and*

$$\exists\, \theta \in [0, \pi^2/L^2) \ \ni \ f_u \ge -\theta.$$

Show that the boundary value problem (4.3) has at least a solution.

4.2 Quasi-linear Elliptic Equations

Let's consider the second order, quasi–linear elliptic equation in non–divergence form

$$a^{ij}(x,u,Du)u_{x_ix_j} + b(x,u,Du) \quad = \quad 0 \qquad \text{in } \Omega, \tag{4.4}$$

with the Dirichlet boundary condition

$$u \quad = \quad g \qquad \text{on } \partial\Omega. \tag{4.5}$$

We make the following assumptions:

- Elliptic assumption:

$$a^{ij}(x,u,p)\xi_i\xi_j > 0 \qquad \forall \xi \in \mathbb{R}^n \setminus \{0\}, x \in \bar{\Omega}, u \in \mathbb{R}, p \in \mathbb{R}^n. \tag{4.6}$$

- Smoothness of coefficients assumption: For some $\alpha \in (0,1)$,

$$a^{ij}, b \quad \in \quad C^{\alpha}(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n). \tag{4.7}$$

- Smoothness of domain and boundary data assumption:

$$\Omega \ \text{ is bounded,} \ \ \partial\Omega \in C^{2+\alpha}, \quad g \in C^{2+\alpha}(\bar{\Omega}). \tag{4.8}$$

- *A priori* estimates: There exist $M > 0$ and $\beta \in (0,1)$ such that for all $\sigma \in [0,1]$, any $C^2(\bar{\Omega})$ solution v to the modified problem

$$\begin{cases} a^{ij}(x,v,Dv)v_{x_ix_j} + \sigma b(x,v,Dv) = 0 & \text{in } \Omega, \\ v = \sigma g & \text{on } \partial\Omega \end{cases} \tag{4.9}$$

  satisfies

$$\|v\|_{C^{1+\beta}(\bar{\Omega})} \leq M. \tag{4.10}$$

Theorem 4.4. *Assume that (4.6), (4.7), (4.8), and (4.10) hold. Then (4.4), (4.5) has at least one solution.*

Remark We do not expect to have uniqueness. Physically, this could represent a steady state limit of a time evolution equation, which can have different limits for different initial conditions. This non-uniqueness is characteristic of interests for nonlinear equations.

Clearly, the most crucial assumption is the a priori estimate. Indeed, the most difficult and important work on PDE theory is a priori estimates.

Proof of Theorem 4.4 Define a map $\mathbf{T} : C^{1+\beta}(\bar{\Omega}) \mapsto C^{2+\alpha\beta}(\bar{\Omega})$ as follows. For each $v \in C^{1+\beta}(\bar{\Omega})$, we set $\mathbf{T}v = u$, the solution to the linear elliptic PDE

$$a^{ij}(x, v(x), Dv(x))u_{x_i x_j} + b(x, v(x), Dv(x)) = 0 \qquad \text{in } \Omega,$$

together with the boundary condition $u = g$ on $\partial\Omega$.

Notice that whenever $v \in C^{1+\beta}(\bar{\Omega})$, we have

$$a^{ij}(x, v(x), Dv(x)), b(x, v(x), Dv(x)) \in C^{\alpha\beta}(\bar{\Omega}).$$

Also, there exists $\lambda = \lambda(\|v\|_{C^{1+\beta}(\bar{\Omega})})$ such that

$$a^{ij}(x, v(x), Dv(x))\, \xi_i \xi_j \geq \lambda \|\xi\|^2 \quad \forall x \in \bar{\Omega},\ \xi \in \mathbb{R}^n.$$

Hence, by the Schauder estimate,

$$\|u\|_{C^{2+\alpha\beta}(\bar{\Omega})} \leq C_0 = C_0(\|v\|_{C^{1+\beta}(\bar{\Omega})}).$$

Thus, setting $\mathbf{X} = C^{1+\beta}(\bar{\Omega})$ we have the following:

1. $\mathbf{T}$ is a compact since $C^{2+\alpha\beta}(\bar{\Omega}) \hookrightarrow\hookrightarrow C^{1+\beta}(\bar{\Omega})$.

2. $\mathbf{T}$ is continuous. Indeed, suupose that $v_m \to v$ in $C^{1+\beta}(\bar{\Omega})$. Set $u_m = \mathbf{T}v_m$. Since

$$\sup_{m \in \mathbb{N}} \|u\|_{C^{2+\alpha\beta}(\bar{\Omega})} < \infty,$$

we have the existence of a subsequence $\{u_{m_j}\}$ with $u_{m_j} \to u$ in $C^2(\bar{\Omega})$. We then have $u = \lim_{m\to\infty} u_m$ exists. Because

$$a^{ij}(x, v_m, Dv_m)u^m_{x_i x_j} + b(x, v_m, Dv_m) = 0 \quad \forall m \in \mathbb{N}$$

$$\Longrightarrow a^{ij}(x, v, Dv)u_{x_i x_j} + b(x, v, Dv) = 0,$$

we see that $u = \mathbf{T}v$. This implies that the whole sequseence $\{u^m\}$ approaches u.

3. There exists a fixed point of $\mathbf{T}$, and hence a solution to (4.4), (4.5). We use the Leray-Schauder fixed point theorem to show this. Set

$$Y = \{v \in \mathbf{X} : \exists \sigma \in [0, 1] \text{ such that } v = \sigma T v\}.$$

Suppose that $v = \sigma T v$. Then $\mathbf{T}v \in C^{2+\alpha\beta}(\bar{\Omega})$, and so is v. In addition, simple substitution shows that v solves (4.9). Hence, by (4.10), we have $\|v\|_{C^{1+\beta}(\bar{\Omega})} \leq M$. Hence, by the Leray-Schauder Fixed Point Theorem, there exists a fixed point of T.

Homework 4.2. *Assume that*

$$a^{ij}(x, u, Du) = a(x, u)\delta_{ij}, \qquad b(x, u, Du) = B(x, Du) - u$$

where a is positive and both a and B are smooth. Show that (4.4) and (4.5) admits at lease a solution.

4.3 The Thermister Problem

Here we describe a physical problem that leads to a system of PDE's to which one can use a linear PDE theory and a fixed point theorem to show the existence [1].

There is an elecrtical device known as a *thermister*, or *thermal resister*. The thermistor is made of special material whose electrical resistivity grows very large with a small change in temperature beyond some desired threshold. Thus, if there is a short-circuit and the current grows very large, the thermister will heat up beyond the threshold temperature and thus the electrical resistivity will be large enough to (automatically) cut off the circuit. When the device cools, the resistivity of the thermister drops, and the circuit will be back to normal operation.

The mathematical model Now we model this physical situation with a system of PDE's. There are two or three spacial dimensions and one temporal dimension. The spacial domain will be represented by Ω, a bounded, simply connected, convex subset of $\mathbb{R}^n$, and the temporal domain is the set of nonnegative reals. Our unkowns will be $u = u(x,t)$, the temperature of the thermister at position x and time t, and $\phi = \phi(x,t)$, the electric potential. Both are real-valued functions. The electrical current is then given by $\vec{\jmath}(x,t) = \sigma\nabla\phi$, where $\sigma = \sigma(u)$ is the electrical conductivity ($\frac{1}{\text{resistivity}}$). We denote by $k = k(u)$ the thermal conductivity so that the heat flux is given by $-k\nabla u$. We shall derive two partial differential equations, one for ϕ and the other for u.

1. From the Ohm's law, the electical current is divergence-free, i.e., $\nabla \cdot \vec{\jmath} = 0$. Hence

$$\nabla \cdot (\sigma(u)\nabla\phi) = 0.$$

Notice that this is an elliptic equation since $\sigma > 0$.

2. Note that the heat source is the Joel's heating whose energy density is given by $\vec{\jmath} \cdot \nabla\phi$. Hence, the conservation of energy gives

$$Cu_t - \nabla \cdot (k(u)\nabla(u)) = \vec{\jmath} \cdot \nabla\phi,$$

equation where C is the specific heat. using the definition of $\vec{\jmath}$, we then obtain

$$Cu_t - \nabla \cdot (k(u)\nabla u) = \sigma(u)|\nabla\phi|^2.$$

This is a parabolic equation.

The boundary conditions We have conditions for both u and ϕ on the boundary of Ω. The condition on u is given by Newton's law of cooling. Thus, for a constant γ that depends on the ambient material, we have

$$k(u)\frac{\partial u}{\partial \vec{n}} = -\gamma u \text{ on } \partial\Omega,$$

where $\vec{n}$ is the unit outward normal to $\partial\Omega$.

[1] This is taken from a research that Dr. Chen did while he was a graduate student.

For ϕ, we have a condition for each of two portions of $\partial\Omega$. We partition $\partial\Omega$ as $\Omega = \Gamma_1 \cup \Gamma_2$, where Γ_1 should be thought of as the "ends" of the thermister where an electrical voltage is applied. We then specify the potential on Γ_1 with

$$\phi = g \text{ on } \Gamma_1 \times \{t \geq 0\}.$$

Finally, the electrical current does not leave the curcuit through the surface Γ_2 so that

$$\frac{\partial \phi}{\partial \vec{n}} = 0 \text{ on } \Gamma_2 \times \{t \geq 0\}.$$

We will be studying the steady-state version of this problem, so there are no initial conditions.

The statement of the steady state problem In the steady state problem, the only change is that $u_t = 0$. Here, we display the problem for easy reference:

$$\begin{aligned} \nabla \cdot (\sigma(u)\nabla\phi) &= 0 && \text{in } \Omega, && (4.11)\\ \nabla \cdot (k(u)\nabla u) &= \sigma(u)|\nabla\phi|^2 && \text{in } \Omega, && (4.12)\\ k(u)\frac{\partial u}{\partial \vec{n}} &= -\gamma u && \text{on } \partial\Omega, && (4.13)\\ \phi &= g && \text{on } \Gamma_1, && (4.14)\\ \frac{\partial \phi}{\partial \vec{n}} &= 0 && \text{on } \Gamma_2. && (4.15) \end{aligned}$$

For convenience, we place assumptions on the quantities σ, k, and g. First, we assume that all of these are smooth. Morever, we assume that σ and k are bounded. Explicitly, we assume that

$$\exists c_0 > 0 \text{ such that } \frac{1}{c_0} \leq \sigma(u), k(u) \leq c_0 \text{ for all } u \in \mathbb{R}. \tag{4.16}$$

Under these assumptions, we have the following

Theorem 4.5 (Existence). *There exists at least one solution (u, ϕ) to this steady state problem.*

Proof Let $\mathbf{X} = C^0(\bar{\Omega})$, and let $M > 0$ to be determined. Define

$$B = \{u\mathbf{X} : \|u\|_{C^0(\bar{\Omega})} \leq M\}.$$

Then B is a closed, bounded, and convex subset of $\mathbf{X}$. Given $u \in B$, we define ϕ by solving (4.11):

$$\begin{aligned} \nabla \cdot (\sigma(u)\nabla\phi) &= 0 && \text{in } \Omega, && (4.17)\\ \phi &= g && \text{on } \Gamma_1, && (4.18)\\ \frac{\partial \phi}{\partial \vec{n}} &= 0 && \text{on } \Gamma_2. && (4.19) \end{aligned}$$

By standard elliptic theory, ϕ is well-defined. In addition, from the Hölder theory, it follows from the boundedness of σ that there exist constants $K = K(c_0, \Omega, \Gamma_1, \Gamma_2)$ and $\alpha = \alpha(c_0) \in (0, 1)$ such that

$$\|\phi\|_{C^\alpha(\bar{\Omega})} \leq K\|g\|_{C^\alpha(\bar{\Omega})} =: K_0. \tag{4.20}$$

Next, we seek u. We find by calculation and the use of (4.11) that (4.12) may be written as

$$\begin{aligned} 0 &= \nabla\cdot(k(u)\nabla u) - 0 - \sigma(u)|\nabla\phi|^2 && (4.21)\\ &= \nabla\cdot(k(u)\nabla u) - (\nabla\cdot(\sigma(u)\nabla\phi))\,\phi - \sigma(u)\nabla\phi\cdot\nabla\phi && (4.22)\\ &= \nabla\cdot(k(u)\nabla u) - \nabla\cdot(\sigma(u)\phi\nabla\phi)\\ &= \nabla\cdot\left[\sigma(u)\left(\frac{k(u)}{\sigma(u)}\nabla u - \nabla\frac{\phi^2}{2}\right)\right]\\ &= \nabla\cdot(\sigma(u)\nabla\psi), && (4.23)\end{aligned}$$

where ψ is defined by

$$\psi = \int_0^u \frac{k(s)}{\sigma(s)}ds - \frac{\phi^2}{2}. \qquad (4.24)$$

If we again simplify the problem by using the new boundary condition [2]

$$\begin{aligned} u &= u_1 \qquad \text{on } \Gamma_1 && (4.25)\\ \frac{\partial u}{\partial \vec{n}} &= u_2 \qquad \text{on } \Gamma_2, && (4.26)\\ & && (4.27)\end{aligned}$$

then we have the following problem in ψ:

$$\begin{aligned} \nabla\cdot(\sigma(u)\nabla\psi) &= o \qquad \text{in } \Omega && (4.28)\\ \psi &= \psi_1 \qquad \text{on } \Gamma_1 && (4.29)\\ \frac{\partial\psi}{\partial\vec{n}} &= \psi_2 \qquad \text{on } \Gamma_1. && (4.30)\end{aligned}$$

Once again, by the Hölder theory, we obtain a constant K_1 such that

$$\|\psi\|_{C^\alpha(\bar\Omega)} \le K_1. \qquad (4.31)$$

Now we define $v = Tu$ by solving implicitly

$$\psi(x) = \int_0^{v(x)} \frac{k(s)}{\sigma(s)}ds - \frac{\phi^2(x)}{2} \qquad \forall x\in\bar\Omega. \qquad (4.32)$$

Thus,

$$\exists K_2 \text{ such that } \|v\|_{C^\alpha(\bar\Omega)} \le K_2.$$

Now we take $M = K_2$. We see that $u \mapsto v = Tu$ maps B into B. The image TB is precompact, and one can show that T is continuous. Hence, by the Shauder fixed Theorem, T has a fixed point, which yields a solution to the steady state thermistor problem.

[2]For the original physical boundary condition, the method described here still work; nevertheless, significant amount of effort on the a priori estimate of ψ is required.

4.4 An Obstacle Problem

The physical problem is to determine what shape an elastic membrane will take when stretched over an obstacle. The idea is that in some places, the membrane will conform to the obstacle, while in other places, the membrane will "stretch out" over valleys.

Let ϕ be the height of the obstacle, considered as a graph over a domain $\Omega \subset \mathbb{R}^n$. Let u be the height of the membrane. We make the observations:

1. The membrane is *over* the obstacle, so we require
$$u \geq \phi.$$
2. To describe the regularity of the membrane when properly above the obstacle, we use
$$u > \phi \Longrightarrow \Delta u = 0.$$
3. We have boundary conditions
$$u = g \qquad \text{on } \partial\Omega.$$
4. Finally, the membrane is not sticky, so it will be everywhere concave up:
$$-\Delta u \geq 0.$$

These coniderations lead to the following obstacle problem
$$\begin{cases} \min\{u-\phi, -\Delta u\} = 0 \text{ a.e. in } \ \Omega \\ u = g \ \text{ on } \partial\Omega. \end{cases} \tag{4.33}$$

Theorem 4.6. *Assume that Ω is bounded with smooth boundary $\partial\Omega$, that g and ϕ are smooth on $\bar{\Omega}$, and $g \geq \phi$. Then the obatacle problem has a unique solution $u \in W^{2,p}(\Omega)$ for any $p \in [1, \infty)$.*

We only sktech the idea of the proof.

Step 1. Solve the related ϵ-problem
$$\begin{aligned} -\Delta u_\epsilon &= \beta_\epsilon(u_\epsilon - \phi) \quad \text{in } \Omega \\ u_\epsilon &= g \qquad \text{on } \partial\Omega. \end{aligned}$$

Here, β_ϵ is something like a push that almost keeps the membrane from going below the surface. A little bit more specifically, β_ϵ is smooth and
$$\beta_\epsilon(u) = 0 \ \text{ for all } \ u \geq 0, \quad \beta'_\epsilon(u) < 0 \ \text{ for all } \ u < 0,$$
$$\lim_{\epsilon \searrow 0} \beta_\epsilon(u) = \infty \quad \forall\, u < 0.$$

The existence of u_ϵ follows by using a Leray–Schauser's fixed point theorem. The uniqueness of u_ϵ follows by a comparison principle, since $\beta_\varepsilon(\cdot)$ is non–increasing.

Step 2. Establish a good estimates bound on the regularity of u_ϵ so that we can extract convergent sequence from $\{u_\epsilon\}_{1\geq\epsilon>0}$.

Set $m = \|\Delta\phi\|_{C^0(\bar\Omega)}$. Let $\alpha_\varepsilon < 0$ be the unique number such that $\beta_\epsilon(\alpha_\epsilon) = m$. We notice that $\alpha_\epsilon \to 0$ as $\epsilon \to 0$.

Since $u_\varepsilon = g \geq \phi$ on $\partial\Omega$, either $u_\epsilon \geq \phi$ on $\bar\Omega$ or there is a negative interior minimum of $u_\epsilon - \phi$. At any of such a minumum, we have $\Delta(u_\epsilon - \phi) \geq 0$ so that $\beta(u_\epsilon - \phi) = -\Delta u_\epsilon \leq m$. That is, $u_\epsilon - \phi \geq \alpha_\epsilon$. Hence,

$$\min_{x\in\bar\Omega}(u_\epsilon - \phi) \geq \alpha_\epsilon.$$

Once we have the lower bounded of $u_\epsilon - \phi$, we then know that

$$0 \leq \beta_\epsilon(u_\epsilon - \phi) \leq \beta_\epsilon(\alpha_\epsilon) = m.$$

Thus

$$|\Delta u_\epsilon| \leq m.$$

An elliptic estimate then show that for any $p \in [1,\infty)$, there exists a positive number M, independent of ε, such that

$$\|u_\epsilon\|_{W^{2,p}(\Omega)} \leq M.$$

Consequently, there exists a function $u \in \cap_{p>1} W^{2,p}(\Omega)$ such that along some sequence $\{\epsilon_j\}_{j=1}^\infty$ satisfying $\epsilon_j \searrow 0$ as $j \to \infty$, there holds $u_{\epsilon_j} \to u$ in $C^{1+\gamma}(\bar\Omega)$ for any $\gamma \in (0,1)$.

It is then an easy exercise to show that u is a solution to the obstacle problem.

Step 4. Uniqueness. Suppose u is a solution in $W^{2,p}(\Omega)$. Then, for any smooth v satisfying $v \geq \phi$ in Ω and $v = g$ on $\partial\Omega$, we have

$$-(u-v)\Delta u \leq 0 \quad \text{in} \quad \Omega$$

since $-\Delta u \geq 0$ and if $u - v > 0$ then $u > \phi$ and $\Delta u = 0$. Thus,

$$0 \geq -\int_\Omega (u-v)\Delta u\, dx = \int_\Omega \nabla u \nabla(u-v)\, dx.$$

Via approximation, the above *variational inequality* holds for all $v \in H^1(\Omega)$ satisfying $v \geq \phi$ in Ω and $v = g$ on $\partial\Omega$.

This leads to a weak formulation of the obstacle problem:

Definition 4.1. *A function $u \in g + H_0^1(\Omega)$ is called a weak solution to the obstacle problem if $u \geq \phi$ and*

$$\int_\Omega \nabla u \nabla(u-v)\, dx \leq 0$$

for all $v \in g + H_0^1(\Omega)$ with $v \geq \phi$.

Now we show that even weak solutions are unique.

Suppose u_1 and u_2 are two weak solutions. Then both u_2 and u_1 can be used as test functions so that

$$\int_\Omega \nabla u_1 \nabla(u_1 - u_2)\, dx \leq 0, \qquad \int_\Omega \nabla u_2 \nabla(u_2 - u_1)\, dx \leq 0.$$

Adding these two inequalities, we obtain

$$\int_\Omega |\nabla(u_1 - u_2)|^2 \, dx \le 0.$$

This implies that $u_1 = u_2$ since both have the same boundary value.

An alternative way to establish the existence of a weak solution is to consider the energy minimizer of

$$E(u) := \int_\Omega |\nabla u|^2 \, dx$$

in the convex set

$$K := \{u \in H^1(\Omega); u = g \text{ on } \partial\Omega, \quad u \ge \phi \text{ in } \Omega\}$$

Theorem 4.7. *Assume that Ω is a bounded domain and $\phi, g \in H^1(\Omega)$ and $g \ge \phi$ in Ω. Then there exists a unique weak solution to the obstacle problem.*

P*Proof.* Denote $e := \inf_{v\in K} E(v)$. Let $\{u_n\}_{n=1}^\infty$ be a energy minizing sequence, i.e., $u_n \in K$ and $\lim_{n\to\infty} E(u_n) = e$. Then for any positive integer n and m,

$$\begin{aligned} \tfrac{1}{2}\int_\Omega |\nabla(u_n - u_m)|^2 &= E(u_n) + E(u_m) - 2E(\tfrac{1}{2}(u_n + u_m)) \\ &\le E(u_n) + E(u_m) - 2e \to 0 \quad \text{as} \quad n, m \to \infty. \end{aligned}$$

Thus $\{u_n\}_{n=1}^\infty$ is a Cauchy sequence in $H^1(\Omega)$, whose limit is in K and is an energy minimizer of $E(\cdot)$ in K.

Suppose u is an energy minimizer. Then for every $v \in K$ and $\theta \in (0,1)$, $v^\theta := (1-\theta)u+\theta v = u + \theta(v - u) \in K$. From $E(u) \le E(v^\theta)$ we then obtain

$$\int_\Omega \nabla u \nabla(u - v) \, dx \le -\theta \int_\Omega |\nabla(v - u)|^2 \, dx \le 0.$$

From this, we see that u is a weak solution and weak solutions are unique.

Remark 4.1. *The disadvantage of the energy minimizer approach is to establish the regularity of the solution. Significantly amount of work is need to show that $u \in W^{2,p}$ for any $p \in [1, \infty)$.*

4.5 A Semilinear Parabolic System

Let $\mathbf{u} = (u_1, \ldots, u_m)$, and consider the problem

$$\begin{aligned} D\mathbf{u}_t &= \Delta\mathbf{u} + \mathbf{F}(\mathbf{u}) && \text{in } \Omega \times (0,\infty), \\ \mathbf{u} &= 0 && \text{on } \partial\Omega \times (0,\infty), \\ \mathbf{u} &= \mathbf{u}_0(x) && \text{on } \Omega \times \{0\}, \end{aligned} \tag{4.34}$$

where D is a diagonal matrix with positive diagonal elements $d_1, \ldots, d_m$. Assume that Ω is a bounded, smooth domain in $\mathbb{R}^n$, and that $\mathbf{F} : \mathbb{R}^m \mapsto \mathbb{R}^m$ is smooth. How can we solve this problem?

If we put $A = D^{-1}\Delta$ and $\mathbf{f} = D^{-1}\mathbf{F}$, then we have the problem

$$\begin{aligned} \frac{d\mathbf{u}}{dt} &= A\mathbf{u} + \mathbf{f}(\mathbf{u}) \\ \mathbf{u}(0) &= \mathbf{u}_0. \end{aligned} \tag{4.35}$$

We set up to consider this as a semigroup action. Define

$$\begin{aligned} \mathbf{X} &= (L^2(\Omega))^m, \text{ and} \\ \mathcal{D}(A) &= (H_2(\Omega) \cap H_0^1(\Omega))^m. \end{aligned}$$

One can show that A generates a semigroup $S(t) = e^{At}$. We then have that the problem (4.35) is equivalent to find $\mathbf{u}$ satisfying

$$\mathbf{u}(t) = e^{At}\mathbf{u}_0 + \int_0^t e^{A(t-\tau)}\mathbf{f}(\mathbf{u}(\tau))d\tau. \tag{4.36}$$

A fixed point problem Let $T > 0$ and $M > 0$ to be determined. Define

$$\mathbf{X} = C(\bar{\Omega} \times [0,T]; \mathbb{R}^m), \quad B = \{\mathbf{u} \in C(\bar{\Omega} \times [0,T]; \mathbb{R}^m) : \|u\|_{C^0} \leq M\}. \tag{4.37}$$

For each $\mathbf{u} \in B$, we define $\mathbf{v} \equiv P\mathbf{u}$ by

$$\mathbf{v} = e^{At}\mathbf{u}_0 + \int_0^t e^{A(t-\tau)}\mathbf{f}(\mathbf{u}(\tau))d\tau. \tag{4.38}$$

We now bound the norm of v as follows, where $\|\cdot\| = \|\cdot\|_{C^0\left((\bar{\Omega}\times[0,T])^m\right)}$ as follows:

$$\begin{aligned} \|v\| &\leq \|e^{At}\| + \int_0^t \|e^{A(t-\tau)} f(\vec{u}(\tau))\| d\tau \\ &\leq c_1\|u_0\| + c_1 t\|f\| \\ &\leq c^\star + c_1 T K(M). \end{aligned}$$

Choose T and M such that

$$M \leq c^\star + c_1 T K(M).$$

(One choice would be $M = 2c^\star; T = \frac{M}{c_1 K(M)}$.) Then P maps B into itself, and one can show that P is compact and thus has a fixed point.

4.6 The Stefan Problem in Solidification

The problem is to model the process by which a solid liquifies or a liquid solidifies. First, we introduce notation to describe the solid and liquid regions. Let Ω represent the container (bounded spacial domain) in which the substance resides. For each time t, we denote the following subsets of Ω:

- Let Ω_t^+ denote the solid region.

- Let Ω_t^- denote the liquid region.
- Assuming that $\partial\Omega_t^+ \cap \Omega = \partial\Omega_t^- \cap \Omega$. We let $\Gamma_t = \partial\Omega_t^\pm \cap \Omega$ denote this interface.

Similarly, we define subsets of $\Omega \times \{t \geq 0\}$:

- Let Q^+ denote the solid phase $\bigcup_{t\geq 0}(\Omega_t^+ \times \{t\})$.
- Let Q^- denote the liquid phase $\bigcup_{t\geq 0}(\Omega_t^- \times \{t\})$.
- Let Γ denote $\bigcup_{t\geq 0}(\Gamma \times \{t\})$.

Our dependent variable will be $\Theta = \Theta(x,t)$, the temperature at position x and time t. Intoduce the physical constants c_S, c_L, k_S, and k_L for the solid and liquid specific heat (c) and thermal conductivity (k). We then have the following models within the solid state Q^+ and liquid state Q^-, respectively:

$$c_S\Theta_t = k_S\Delta\Theta; \tag{4.39}$$

$$c_L\Theta_t = k_L\Delta\Theta. \tag{4.40}$$

Now we model the interfacial conditions. The first is just that the temperature be zero on the interface. That is,

$$\Theta = 0 \quad \text{on } \Gamma. \tag{4.41}$$

The other condition relates the motion of Γ_t to the jump of temperature flux across the boundary Γ_t. If Ω is one dimensional, and if Γ_t is then just a single point $s(t)$, we have

$$\left[k_L\frac{\partial\Theta^-}{\partial x} - k_S\frac{\partial\Theta^+}{\partial x}\right] = l\frac{ds}{dt}, \tag{4.42}$$

where l is the constant latent heat of the substance. (This is a measure of how much energy it takes to melt one unit mass of the substance at equillibrium temperature.) In general, this becomes

$$\left[k_L\frac{\partial\Theta^-}{\partial\vec{n}} - k_S\frac{\partial\Theta^+}{\partial\vec{n}}\right] = l \times (\text{normal velocity of } \Gamma_t). \tag{4.43}$$

The weak formulation. We study what the entropy $\eta = \eta(\Theta)$ must satisfy in this problem. First, we consider the thermal conductivity as a funtion of Θ by simply defining

$$k(\Theta) = \begin{cases} k_S & \text{if } \Theta < 0 \\ \frac{1}{2}(k_S + k_L) & \text{if } \Theta = 0 \\ k_L & \text{if } \Theta > 0. \end{cases}$$

Next, we define the enthalpy. Notice that the enthalpy is either multivalued or not well-defined as a function of Θ when $\Theta = 0$, the equillibrium temperature. We require

$$\eta(x,t) \begin{cases} = c_S\Theta & \text{if } \Theta < 0 \\ \in [0,l] & \text{if } \Theta = 0 \\ = l + c_L\Theta & \text{if } \Theta > 0. \end{cases}$$

One can then show that

$$\eta_t = \nabla \cdot (k(\Theta)\nabla\Theta)$$

if Θ is a weak solution to this problem.

In order to simplify this expression slightly, we make the change of variables

$$v = \begin{cases} k_S\Theta & \text{if} \quad \Theta < 0 \\ k_L\Theta & \text{if} \quad \Theta < 0. \end{cases}$$

Then we express the enthalpy with a fuction σ of v:

$$\sigma(v) = \begin{cases} \alpha_S v & \text{if} \quad v < 0 \\ \alpha_L v + \lambda & \text{if} \quad v > 0, \end{cases}$$

where

$$\begin{aligned} \alpha_S &= \frac{c_S}{k_S}, \\ \alpha_L &= \frac{c_L}{k_L}, \text{ and} \\ \lambda &= \frac{l}{k_L}. \end{aligned}$$

This leads to the following problem in v:

$$\frac{\partial}{\partial t}\sigma(v) = \Delta v. \tag{4.44}$$

Definition 4.2. *A pair (u, γ) is said to be* a weak solution to (4.44) *if $(u, \gamma) \in (L^2(\Omega \times (0,T)))^2$,*

$$\int_0^T \int_\Omega (\gamma\phi_t + u\Delta\phi)\, dx\, dt = 0 \qquad \forall \phi \in C_0^\infty(\Omega \times (0,T]),$$

and

$$\begin{aligned} \gamma(x,t) &= \sigma(u(x,t)) \text{ when } u(x,t) \neq 0; \\ \gamma(x,t) &\in [0,\lambda] \text{ when } u(x,t) = 0. \end{aligned}$$

An initial-boundary value problem. Putting initial conditions on the entropy and boundary conditions on the temerature leads to the following problem:

$$\begin{aligned} \gamma_t &= \Delta u && \text{in } \Omega \times (0,\infty) \\ \gamma &= \gamma_0 && \text{on } \Omega \times \{t = 0\} \\ u &= g && \text{on } \partial\Omega \times [0,\infty). \end{aligned} \tag{4.45}$$

Notice that our definition of weak solution to this problem implies that for a weak solution of (4.45) we must have (integrating by parts) that

$$\int_0^\infty \int_\Omega (\gamma\phi_t + v\Delta\phi) = \int_0^\infty \int_{\partial\Omega} g\frac{\partial\phi}{\partial\vec{n}}\, d\vec{S}\, dt - \int_\Omega \gamma_0(x)\phi(x,0)\, dx$$

for every $\phi \in C^2(\Omega \times (0,\infty))$ that satisfies

$$\begin{aligned} \phi &= 0 && \text{on } \partial\Omega \times (0,\infty), \text{ and} \\ \phi &= 0 && \text{on } \Omega \times [T,\infty) \text{ for some } T > 0. \end{aligned}$$

To prove the existence of a weak solution, we use the following steps.

Step 1: A mollified problem. This equation is degenerate. We therefore study a "mollified" version of this problem. Consider, for $\epsilon > 0$, the problem

$$\begin{aligned} (\sigma_\epsilon'(u^\epsilon))_t &= \Delta u^\epsilon \qquad \text{in } \Omega \times (0,\infty) \\ u^\epsilon &= g_\epsilon \qquad \text{on } (\Omega \times \{t=0\}) \cup (\partial\Omega \times [0,\infty)) \,. \end{aligned} \tag{4.46}$$

We define the "mollified" data so as to satisfy the following requirements:

1. The function σ_ϵ is a smooth approximation to σ. In particular, we require that

$$\begin{aligned} \sigma_\epsilon(u) &= \sigma(u) && \text{for} |u| > \epsilon, \\ \sigma_\epsilon'(u) &> \min\{\alpha_L, \alpha_C\} && \text{for} |u| \le \epsilon, \\ \sigma_\epsilon'(u) &\le \tfrac{2\lambda}{\epsilon} && \text{for} |u| \le \epsilon, \\ \sigma_\epsilon &\in C^2(\mathbb{R}), && \\ \lim_{\epsilon\to 0} \sigma_\epsilon(u) &= \sigma(u) && \forall u \neq 0. \end{aligned}$$

2. We require that $g_\epsilon \in C^\infty(\partial\Omega \times [0,\infty))$, and that $g_\epsilon \to g$ as $\epsilon \to 0$.

3. For initial condition compatability, we require that $\sigma_\epsilon(g_\epsilon) \to \gamma_0$ as $\epsilon \to 0$.

Existence of a solution to the "ϵ-problem." We have

$$u_t = \frac{1}{\sigma_\epsilon'(u)} \Delta u.$$

Let $T > 0$ be fixed. For each $u \in C^0(\bar\Omega \times [0,T])$, define $v \equiv Pu$ as the solution to

$$\begin{aligned} v_t &= \tfrac{1}{\sigma_\epsilon'(u)} \Delta v && \text{in } \Omega \times (0,T) \\ v &= g_\epsilon && \text{on } (\partial\Omega \times (0,T)) \cup (\Omega \times \{t=0\}). \end{aligned}$$

By the Hölder estimate, there are constants $\beta_\epsilon \in (0,1)$ and $M_\epsilon > 0$ such that

$$\|v\|_{C^{\beta_\epsilon, \frac{\beta_\epsilon}{2}}(\bar\Omega\times[0,T])} \le M_\epsilon = M_\epsilon(\|g_\epsilon\|_{C^{\beta_\epsilon}(\bar\Omega\times[0,T])},\ \min \sigma_\epsilon',\ \max \sigma_\epsilon') > 0.$$

One can show that $P : C^0 \mapsto C^0$ is compact and continuous and maps

$$B = \left\{ u : \|u\|_{C^{\beta_\epsilon, \frac{\beta_\epsilon}{2}}(\bar\Omega\times[0,T])} \le M_\epsilon \right\}$$

into itself. Thus P has a fixed point, which gives the existence of a solution u^ϵ to the ϵ problem. and the Schauder estimate improves the smoothness to $u^\epsilon \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\bar\Omega \times [0,T])$.

Uniqueness of the "ϵ-problem" solution. Suppose that u_1 and u_2 are solutions of (4.46). By the above estimate, we have $u_1, u_2 \in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar\Omega \times [0,T])$, and we can write

$$(\sigma_\epsilon(u_1) - \sigma_\epsilon(u_2))_t - \Delta(u_1 - u_2) = 0.$$

We multiply this expression by $\int_t^T (u_1-u_2)(x,\tau)\,d\tau$ and integrate over $\Omega\times[0,T]$ to obtain

$$\begin{aligned} 0 &= \int_0^T\int_\Omega \Big(\{(\sigma_\epsilon(u_1)-\sigma_\epsilon(u_2))_t - \Delta(u_1-u_2)\} \\ &\qquad \int_t^T (u_1-u_2)(x,\tau)\,d\tau \Big)\,dx\,dt \\ &=: I - II, \end{aligned}$$

where the definitions of I and II are explicit below. Now we have, using integration by parts in t, that

$$\begin{aligned} I &= \int_0^T\int_\Omega \Big((\sigma_\epsilon(u_1)-\sigma_\epsilon(u_2))_t \int_t^T (u_1-u_2)(x,\tau)\,d\tau \Big)\,dx\,dt \\ &= \left[\int_\Omega (\sigma_\epsilon(u_1)-\sigma_\epsilon(u_2)) \int_t^T (u_1-u_2)(x,\tau)\,d\tau\,dx \right]_{t=0}^{t=T} \\ &\quad - \int_0^T\int_\Omega (\sigma_\epsilon(u_1)-\sigma_\epsilon(u_2))(-(u_1-u_2))\,dx\,dt \\ &=: I_1 - I_2. \end{aligned}$$

Notice that when $t=T$, the integral within I_1 vansishes, and when $t=0$, the other factor in I_1 vanishes since u_1 and u_2 then agree. Thus, $I_1=0$. Notice also that since σ_ϵ is monotone, the integrand of I_2 is non-positive. This shows that I is non-negative.

Next, we calculate that

$$\begin{aligned} II &= \int_0^T\int_\Omega \Big(\Delta(u_1-u_2) \int_t^T (u_1-u_2)(x,\tau)\,d\tau \Big)\,dx\,dt \\ &= -\int_0^T\int_\Omega \Big(\nabla(u_1-u_2)\,\nabla\!\int_t^T (u_1-u_2)(x,\tau)\,d\tau \Big)\,dx\,dt \\ &= -\int_\Omega\int_0^T \Big(\nabla(u_1-u_2) \int_t^T \nabla(u_1-u_2)(x,\tau)\,d\tau \Big)\,dt\,dx, \end{aligned}$$

by integration by parts. Notice that the boundary term vanished since $u_1=u_2$ on $\partial\Omega$. This, in turn is equal to

$$\begin{aligned} \ldots &= \frac{1}{2}\int_\Omega\int_0^T \frac{d}{dt}\left(\left(\int_t^T \nabla(u_1-u_2)(x,\tau)\,d\tau \right)^2 \right) dt\,dx \\ &= \frac{1}{2}\int_\Omega \left[\left(\int_t^T \nabla(u_1-u_2)(x,\tau)\,d\tau \right)^2 \right]_{t=0}^{t=T} dx \\ &= -\frac{1}{2}\int_\Omega \left(\int_0^T \nabla(u_1-u_2)(x,\tau)\,d\tau \right)^2 dx. \end{aligned}$$

Thus, we also have that $-II$ is nonnegative. Recall that $I-II=0$; thus $I=II=0$. This givs the desired $u_1=u_2$. This is because these functions are smooth, agree on the boundary of Ω, and (since $II=0$) $\nabla(u_1-u_2)=0$.

Step 2: Existence of a solution to the main problem. We wish to show that the solutions to the ϵ-problems converge to a solution to the main problem. Consider the family $\{u^\epsilon\}_{0<\epsilon\leq 1}$ of solutions to the ϵ-problems.

1. By the maximum principle,

$$\|u^\epsilon\|_{L^\infty} \leq \|g^\epsilon\|_{L^\infty} \leq M, \tag{4.47}$$

assuming that g itself is bounded.

2. Multiplying $\sigma'(u^\epsilon)u_t^\epsilon = \Delta u^\epsilon$ by u_t^ϵ, we have, for every $t > 0$ that

$$\begin{aligned} 0 &= \int_0^t \int_\Omega \sigma'(u^\epsilon)(u_t^\epsilon)^2 \, dx \, dt - \int_0^t \int_\Omega u_t^\epsilon \Delta u_t^\epsilon \, dx \, dt \\ &=: J + K. \end{aligned}$$

We have for J that

$$J \geq \|\sigma'\|_\infty \int_0^t \int_\Omega (u_t^\epsilon)^2 \, dx \, dt.$$

And for K,

$$\begin{aligned} K &= -\int_0^t \int_{\partial\Omega} u_t^\epsilon \frac{\partial u^\epsilon}{\partial \vec{n}} \, dx \, d\tau + \int_0^t \int_\Omega \nabla u_t^\epsilon \nabla u_t^\epsilon \, dx \, d\tau \\ &= \int_0^t \frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla u^\epsilon|^2 \, dx - \int_0^t \int_{\partial\Omega} u_t^\epsilon \frac{\partial u^\epsilon}{\partial \vec{n}} \, dx \, d\tau \\ &\geq \frac{1}{2}\left[\int_\Omega |\nabla u^\epsilon(x,t)|^2 \, dx - \int_\Omega |\nabla g^\epsilon|^2 \, dx\right] \\ &\quad - \int_0^t \int_\Omega |g_t^\epsilon||\nabla u^\epsilon| \, dx \, d\tau \end{aligned}$$

Hence,

$$\begin{aligned} \frac{1}{2}\int_\Omega |\nabla u^\epsilon(x,t)|^2 \, dx \quad &+ \quad \sup\{\sigma'\} \int_0^t \int_\Omega (u_t^\epsilon)^2 \, dx \, dt \\ &\leq \frac{1}{2}\int_\Omega |\nabla g^\epsilon(x,0)|^2 \, dx + \int_0^t \int_\Omega |g_t^\epsilon|||\nabla u_t^\epsilon| \, dx \, dt. \end{aligned}$$

(By a parabolic estimate, $|\nabla u^\epsilon| \leq C$ on $\partial\Omega \times [0,T]$.)

$$\therefore \sup_{0\leq t\leq T} \int_\Omega |\nabla u^\epsilon(x,t)|^2 \, dx + \int_0^T \int_\Omega (u_t^\epsilon)^2 \, dx \, dt \leq C(T),$$

independently of ϵ.

From 1 and 2, we can find a sequence $\{\epsilon_j\}_{j=1}^{\infty}$ such that $\epsilon_j \to 0$ and functions u and γ such that

$$u^{\epsilon_j} \to u \quad \text{in} \quad L^2(\Omega \times (0,T)),$$

(by **2**, the family u^ϵ is bounded in the compact subset $H_0^1(\Omega \times [0,T])$) and

$$\sigma_\epsilon(u^\epsilon) \rightharpoonup \gamma \quad \begin{cases} = \alpha_L + \lambda & \text{if} \quad u > 0, \\ \in [0,\lambda] & \text{if} \quad u = 0, \\ = \alpha_S u & \text{if} \quad u < 0, \end{cases} \qquad a.e.(\Omega \times (0,T)).$$

Now for any smooth test function $\phi \in C^{2,1}(\bar{\Omega}\times[0,T])$ with $\phi = 0$ on $(\partial\Omega \times [0,T]) \cup (\Omega \times \{t = T\})$, we have

$$\begin{aligned} \int_0^T \int_\Omega (\sigma_\epsilon(u^\epsilon)\phi_t + u^\epsilon \Delta\phi)\, dx\, dt &= \int_0^T \int_{\partial\Omega} g^\epsilon \frac{\partial \phi}{\partial \vec{n}}\, d\vec{s}\, dt \\ &- \int_\Omega \sigma_\epsilon(g^\epsilon(x,0))\phi(x,0)\, dx. \end{aligned}$$

Let $\epsilon = \epsilon_j \searrow 0$. We have, from te weak convergence of $\sigma_\epsilon(u^\epsilon)$ and convergence of all other quantities, that

$$\int_0^T \int_\Omega (\gamma\phi_t + u\Delta\phi)\, dx\, dt = \int_0^T \int_{\partial\Omega} g \frac{\partial\phi}{\partial\vec{n}}\, d\vec{s}\, dt - \int_\Omega \gamma_0 \phi(x,t)\, dx.$$

Hence, (u,γ) is a weak solution. In addition, we have the bounds

$$\begin{aligned} \|u\|_{L^\infty} &\le C, \text{ and} \\ \|u_t\|_{L^2(\Omega\times[0,T])} + \sup_{0\le t\le T} \|\nabla u(\cdot,t)\|_{L^2(\Omega)} &\le C(T). \end{aligned}$$

Step 3: Uniqueness of this solution Suppose that (u_1,γ_1) and (u_2,γ_2) are two solutions. Then for any smooth test function ϕ that vanishes on $(\partial\Omega \times [0,T]) \cup (\Omega \times \{t = T\})$, we have

$$\begin{aligned} 0 &= \int_0^T \int_\Omega (\gamma_1 - \gamma_2)\phi_t + (u_1 - u_2)\Delta\phi)\, dx\, dt \\ &= \int_0^T \int_\Omega (\gamma_1 - \gamma_2)(\phi_t + e\Delta\phi)\, dx\, dt, \end{aligned}$$

where

$$e = \begin{cases} \frac{u_1 - u_2}{\gamma_1 - \gamma_2} & \text{if } u_1 \ne u_2 \\ 0 & u_1 = u_2. \end{cases}$$

We take $\phi = \phi^m$, where ϕ^m is the solution to

$$\begin{aligned} \phi_t^m + e^m \Delta\phi^m &= f^m \quad \text{in } \Omega \times [0,T) \\ \phi^m &= 0 \quad \text{on } (\partial\Omega \times [0,T]) \cup (\Omega \times \{t = T\}), \end{aligned}$$

where e^m and f^m are smooth functions to be determined. We now have

$$\begin{aligned} 0 &= \int_0^T \int_\Omega (\gamma_1 - \gamma_2)(\phi_t^m + (e - e^m)\Delta\phi^m + e^m\Delta\phi^m)\, dx\, dt \\ &= \int_0^T \int_\Omega (\gamma_1 - \gamma_2) f^m\, dx\, dt + \int_0^T \int_\Omega (e - e^m)\Delta\phi^m\, dx\, dt \\ &=: J_1^m + J_2^m. \end{aligned}$$

For the second integral, we see that

$$|J_2^m| \le \sqrt{\int_0^T \int_\Omega \frac{(e-e^m)^2}{e^m}\,dx\,dt}\sqrt{\int_0^T \int_\Omega e^m(\Delta\phi^m)^2\,dx\,dt.}$$

Also, since

$$\left(\frac{1}{\sqrt{e^m}}\phi_t^m + \sqrt{e^m}\Delta\phi^m\right)^2 = \left(\frac{f^m}{\sqrt{e^m}}\right)^2,$$

we can derive that

$$\int_0^T \int_\Omega e^m(\Delta\phi^m)^2\,dx\,dt \le \int_0^T \int_\Omega \frac{(f^m)^2}{e^m}\,dx\,dt.$$

Now we want to find appropriate e^m and f^m so that we can use this to deduce that $u_1 = u_2$. Take ϵ_m small enough that

$$\|\rho_{\epsilon_m} * e - e\|_{L^2} \le m^{-2}.$$

Note that

1. e is bounded, since σ' is bounded.

2. $\rho_{\epsilon_m} * e \ge 0$, since $e \ge 0$.

Now take

$$e^m \equiv \rho_{\epsilon_m} * e + \frac{1}{m}.$$

Then we obtain

$$\int_0^T \int_\Omega \frac{(e-e^m)^2}{e^m}\,dx\,dt = \int_0^T \int_\Omega \frac{\left(\frac{1}{m} - (\rho_{\epsilon_m} * e - e)\right)^2}{e^m}\,dx\,dt \le \frac{2}{m^2}.$$

Hence, if we take

$$f^m \equiv \rho_{\epsilon_m} * (u_1 - u_2),$$

then

$$|J_2| \le \sqrt{\frac{2}{m}} \int \frac{(f^m)^2}{e^m} \to 0 \text{ as } m \to \infty.$$

So, finally, if we let $m \to \infty$, we get

$$0 \ge \lim_{m\to\infty} J_1^m = \int_0^T \int_\Omega (\gamma_1 - \gamma_2)(u_1 - u_2)\,dx\,dt,$$

which, by the monotonicity of γ in u, shows that $u_1 = u_2$ a.e.