EXISTENCE AND EXACT MULTIPLICITY OF PHASELOCKED SOLUTIONS OF A MODEL OF MUTUALLY COUPLED OSCILLATORS

WILLIAM C. TROY*

Abstract. We investigate the existence and exact multiplicity of phaselocked solutions of a system of coupled oscillators. Under general assumptions on the form of frequency distribution, we derive new, easily verified criteria that guarantee that either (i) exactly one solution exists, or (ii) exactly two solutions coexist over an entire interval of values of the key parameter \( \gamma \). We illustrate our results with an example in which each of these possibilities occurs. Problems for future research are suggested.

Key words. integral equation, phaselocking, coupled system polynomial

AMS subject classifications. 34AXX, 45GXX


\[
\theta_j'(t) = \omega_j + \alpha \sum_{k=1}^{N} a_{jk}(\theta_k - \theta_j), \quad j = 1, \ldots, N
\]  

(1.1)

where \( \theta_j \) is the phase of the \( j \)-th oscillator, \( \omega_j \) is its uncoupled frequency, and \( a_{jk}(u) \) is a periodic function representing coupling. System (1.1), also known as a Kuramoto model [6], was used by Ermentrout and Kopell [5] to model frequency plateaus in mammalian intestine. Cohen et al [3] used similar systems to model swimming in fish. In recent studies, various forms of (1.1) have been used to model the onset of synchronization in diverse settings [1, 2, 6, 8, 9, 12]. In these studies simplifying assumptions are needed to make the analysis mathematically tractable. For example, in their model of Chimera state synchronization, Abrams et al [1] assume that the frequencies are not random, and all have the same value. In other settings, where frequencies are random [1, 2, 6, 8, 9, 12], mathematical tractability requires the use of Cauchy-Lorentzian distributions to model frequency distribution.

In this paper we follow Ermentrout’s alternative approach [4] to study synchronization in (1.1) which allows more flexibility in the choice of frequency distribution. He assumes that each oscillator is coupled to every other one by \( a_{jk}(u) = \frac{\sin(u)}{u} \) (the use of \( \sin(u) \) as coupling function dates back to Turing [13]). Ermentrout derives a nonlinear equation ((1.10) and its equivalent formulation (1.18) below) which is a criterion for the existence of phaselocked solutions of (1.1). In order to accurately state our goals and main result (Theorem 1.1), we first need to give a brief description of the two step process Ermentrout follows in deriving (1.10) and (1.18).

Step I. First, he assumes that the frequencies of the oscillators are not randomly distributed, and takes the the continuum limit of (1.1) as \( N \to \infty \), to obtain the integro-differential equation

\[
\frac{\partial \theta}{\partial t} = \omega(x) + \alpha \int_{0}^{1} \sin (\theta(x',t) - \theta(x,t)) \, dx',
\]  

(1.2)
where \( t \geq 0, \ 0 \leq x \leq 1 \) and \( \alpha > 0 \). He assumes that the frequency function \( \omega(x) \) is integrable, and that

\[
\omega(x) = \bar{\omega} + \bar{\gamma} \Delta(x), \quad \sup_{0 \leq x \leq 1} |\Delta(x)| = 1, \quad \bar{\gamma} = \sup_{0 \leq x \leq 1} |\omega(x) - \bar{\omega}| > 0.
\]

Thus, the space averaged equation is

\[
\frac{d\bar{\theta}}{dt} = \bar{\omega}, \quad \bar{\omega} = \int_0^1 \omega(x)dx, \quad \bar{\theta}(t) = \int_0^1 \theta(x,t)dx.
\]

An important conclusion from (1.3) and (1.4) is that

\[
\int_0^1 \Delta(x) = 0.
\]

Ermentrout looks for phase-locked solutions of (1.2) which have the form

\[
\theta(x,t) = \bar{\omega}t + \phi(x).
\]

Substituting (1.3) and (1.6) into (1.2) gives

\[
\gamma \Delta(x) = \int_0^1 \sin(\phi(x) - \phi(x'))dx',
\]

where \( \gamma = \frac{\bar{\gamma}}{\alpha} \). A non-trivial phase-locked solution exists if there is a \( \phi(x) \) which satisfies (1.7) for some \( \gamma > 0 \). It follows from (1.7), and the restriction \( \sup_{0 \leq x \leq 1} |\Delta(x)| = 1, \) that \( 0 < \gamma \leq 1 \). Next, observe that (1.7) is equivalent to

\[
\gamma \Delta(x) = \sin(\phi(x)) \int_0^1 \cos(\phi(x'))dx' - \cos(\phi(x)) \int_0^1 \sin(\phi(x'))dx'.
\]

Ermentrout claims, on the basis of numerical simulations, that stable phase-locked solutions exist when

\[
\gamma \Delta(x) = C \sin(\phi(x)).
\]

Combining (1.5) with (1.9) gives \( \int_0^1 \sin(\phi(x'))dx' = 0 \), and therefore (1.8) reduces to

\[
C = \int_0^1 \sqrt{1 - \frac{\gamma^2}{\alpha^2} \Delta^2(x')}dx'.
\]

It follows from (1.9), and the restriction \( \sup_{0 \leq x \leq 1} |\Delta(x)| = 1 \), that \( C \geq \gamma \). Also, we conclude from (1.10) that \( C \leq 1 \). Thus, the range of \( \gamma \) and \( C \) is

\[
0 < \gamma \leq 1 \quad \text{and} \quad \gamma \leq C \leq 1.
\]

If \( (\gamma, C) = \left( \frac{\bar{\gamma}}{\alpha}, C \right) \) satisfies (1.10)-(1.11) then it follows from (1.3) and (1.9) that

\[
\phi(x) = \sin^{-1} \left( \frac{\gamma \Delta(x)}{C} \right) = \sin^{-1} \left( \frac{\omega(x) - \bar{\omega}}{\alpha C} \right),
\]
and the corresponding phaselocked solution (1.6) of (1.2) is

$$\theta(x, t) = \bar{\omega}t + \sin^{-1} \left( \frac{\omega(x) - \bar{\omega}}{\alpha C} \right).$$

**Step II.** Ermentrout [4] recasts (1.10) in terms of a probabilistic model. For this he assumes that the frequencies are randomly distributed and satisfy

$$\omega_j = \bar{\omega} + \gamma Z_j, \quad j = 1, \ldots, N$$

where the $Z_j$'s are independent, identically distributed random variables, with range $-1 \leq Z_j \leq 1$ and common pdf $f(z)$. He assumes

**(A0) $f(z)$ is continuous and symmetric on $[-1,1]$, and non-increasing on $[0,1]$.**

Thus, the mean and variance of $Z_j$ satisfy

$$E(Z_j) = \int_{-1}^{1} zf(z)dz = 0 \quad \text{and} \quad \text{Var}(Z_j) = \int_{-1}^{1} z^2 f(z)dz \leq 1, \quad j = 1, \ldots, N$$

It follows from (1.14) and (1.15) that

$$E(\omega_j) = \bar{\omega} \quad \text{and} \quad \text{Var}(\omega_j) = \gamma^2 \int_{-1}^{1} z^2 f(z)dz \leq \gamma^2, \quad j = 1, \ldots, N$$

Next, Ermentrout considers the discrete equation

$$C = \frac{1}{N} \sum_{j=1}^{N} \sqrt{1 - \gamma^2 C^2 Z_j^2},$$

and uses the law of large numbers to prove that, as $N \to \infty$, (1.17) converges to

$$C = \int_{0}^{1} \sqrt{1 - \frac{\gamma^2}{C^2} z^2} f(z)dz.$$  

Assumption (1.24) implies that (1.18) reduces to (1.10). To see this, let

$$x = \int_{-1}^{z} f(\tau)d\tau, \quad -1 \leq z \leq 1.$$  

It follows from (1.24) that $x(z)$ is an increasing function of $z \in [-1,1]$. Therefore $x(z)$ is invertible and there is a unique function $\Delta(x)$ such that

$$z = \Delta(x(z)), \quad -1 \leq z \leq 1.$$  

Substituting (1.19)-(1.20) into (1.18) gives

$$C = \int_{0}^{1} \sqrt{1 - \frac{\gamma^2}{C^2} z^2} f(z)dz = \int_{0}^{1} \sqrt{1 - \frac{\gamma^2}{C^2} \Delta^2(x)}dx.$$  

It follows from (1.21) that synchronization criteria (1.10) and (1.18) are equivalent.
When \( f(z) \) satisfies (A_0), Ermentrout shows that there is at least one solution of (1.10) and (1.18) for each \( \gamma \in (0, \gamma^*) \), where

\[
\gamma^* = \int_{-1}^{1} \sqrt{1 - z^2} f(z) dz = \int_{0}^{1} \sqrt{1 - \Delta^2(x)} dx.
\]

He claims, based on numerical simulation, that his solutions are stable. An important step towards proving stability is to determine whether solutions of (1.10) and (1.18) are unique. However, he does not investigate uniqueness. Instead his main focus is to show that, if \( f(z) \) satisfies (A_0), then \( \gamma^* \) is the “phaselocking threshold,” the largest \( \gamma \in (0, 1) \) where a solution can exist.

Our purpose is to give the first complete proof of existence, and exact multiplicity, of solutions of phaselocking criteria (1.10) and (1.18). Our assumptions on \( f(z) \) and \( \Delta(x) \) (see (A_1)-(A_3) below) include (A_0), but are significantly wider ranging than (A_0). In particular, we remove the requirement that \( f(z) \) is non-increasing on \([0, 1]\) (see (1.24) below). We derive two new, general criteria (inequalities (1.32) and (1.33) in Theorem 1.1) which guarantee that either (I) exactly one solution of (1.10) and (1.18) exists, or (II) exactly two solutions coexist, over an entire interval of values of the parameter \( \gamma \). Our most novel result is the prediction that two solutions coexist when criterion (1.33) holds. When two solutions coexist we prove that (i) \( f(z) \) lies outside the range of Ermentrout’s assumption (A_0), and (ii) the phaselocking threshold is greater than the value \( \gamma^* \) predicted in [4] (see the example in Section 2).

We focus on analyzing the function

\[
H(\gamma, C) = C - \int_{0}^{1} \sqrt{1 - \frac{\gamma^2}{C^2} \Delta^2(x')} dx'.
\]

Phaselocked solutions exist when \( H(\gamma, C) = 0 \).

**Assumptions.** We assume the following:

(A_1) \( \Delta(x) \) is derived from a pdf \( f(z) \) satisfying

\[
(1.24) f(z) \geq 0, \text{ continuous and symmetric on } [-1, 1]; \quad f''(z) > 0 \text{ if } f(z) = f'(z) = 0
\]

**Remark.** The pdf \( f(z) \) can oscillate multiple times on \([-1, 1]\).

(A_2) \( \Delta(x) \) is continuous and increasing on \([0, 1]\), satisfies (1.3)-(1.5), and

\[
\gamma^* = \int_{0}^{1} \sqrt{1 - \Delta^2(x)} dx > 0.
\]

(A_3) \( H(\gamma, C), \frac{\partial H}{\partial \gamma}(\gamma, C), \frac{\partial H}{\partial C}(\gamma, C) \) and \( \frac{\partial^2 H}{\partial C^2}(\gamma, C) \) are continuous over parameter range (1.11), where they satisfy

\[
(1.26) \quad \frac{\partial H}{\partial C}(\gamma, C) = 1 - \frac{\gamma^2}{C^2} \int_{0}^{1} \frac{\Delta^2(x')}{(1 - \frac{\gamma^2}{C^2} \Delta^2(x'))^{1/2}} dx,
\]

\[
(1.27) \quad \frac{\partial H}{\partial \gamma}(\gamma, C) = \frac{\gamma}{C^2} \int_{0}^{1} \frac{\Delta^2(x')}{(1 - \frac{\gamma^2}{C^2} \Delta^2(x'))^{1/2}} dx > 0.
\]
\( \frac{\partial^2 H}{\partial C^2}(\gamma, C) = \frac{3\gamma^2}{C^4} \int_0^1 \frac{\Delta^2(x')}{(1 - \gamma^2 \Delta^2(x'))^{1/2}} dx + \frac{\gamma^4}{C^6} \int_0^1 \frac{\Delta^4(x')}{(1 - \gamma^2 \Delta^2(x'))^{3/2}} dx > 0. \)

Before stating our main result we make three important observations. First, it follows from (1.3), (1.23), and assumptions \((A_1)-(A_3)\) that

\[
H(\gamma, \gamma) \begin{cases} < 0, & 0 < \gamma < \gamma_* \\ = 0, & \gamma = \gamma_* \\ > 0, & \gamma_* < \gamma \leq 1, \end{cases}
\]

where \(\gamma_*\) satisfies (1.25). Property (1.29) plays an important role in the proof of our main result. Second, because \(H(\gamma_*\gamma_*) = 0\), we conclude that a phaselocked solution \(\theta_*(x, t)\) of (1.2) exists at \((\gamma, C) = (\gamma_*, \gamma_*)\), and is given by

\[
\theta_*(x, t) = \bar{\omega}t + \phi_*(x) = \sin^{-1} \left( \frac{\omega(x) - \bar{\omega}}{\alpha \gamma_*} \right).
\]

This property also plays an important role in the proof of our main result. Third, combine (1.25) with (1.27) and get

\[
\frac{\partial H}{\partial C}(\gamma_*, \gamma_*) = \frac{1}{\gamma_*} \int_0^1 \frac{1 - 2\Delta^2(x)}{\sqrt{1 - \Delta^2}} dx.
\]

**Theorem 1.1.** Let \(f(z), \Delta(x), \gamma_*\) and \(H(\gamma, C)\) satisfy (1.24), (1.23) and assumptions \((A_1)-(A_3)\).

**I** Suppose that

\[
\frac{\partial H}{\partial C}(\gamma_*, \gamma_*) = \frac{1}{\gamma_*} \int_0^1 \frac{1 - 2\Delta^2(x)}{\sqrt{1 - \Delta^2}} dx \geq 0.
\]

Then \(\gamma_*\) is the phaselocking threshold, and the following properties hold:

(i) if \(0 < \gamma \leq \gamma_*\) then there is a unique solution of \((1.10)\);

(ii) if \(\gamma_* < \gamma \leq 1\) then there is no solution \((1.10)\);

**II** Suppose that

\[
\frac{\partial H}{\partial C}(\gamma_*, \gamma_*) = \frac{1}{\gamma_*} \int_0^1 \frac{1 - 2\Delta^2(x)}{\sqrt{1 - \Delta^2}} dx < 0.
\]

Then there is a value \(\gamma^* > \gamma_*\) such that \(\gamma^*\) is the phaselocking threshold, and the following properties hold:

(iii) if \(0 < \gamma < \gamma_*\) then there is a unique solution of \((1.10)\);

(iv) if \(\gamma_* \leq \gamma < \gamma^*\) then there are exactly two solutions of \((1.10)\);

(v) if \(\gamma = \gamma^*\) then there is exactly one solution of \((1.10)\);

(vi) if \(\gamma^* < \gamma \leq 1\) then there is no solution of \((1.10)\);

There is a wide diversity of functions \(f(z)\) and \(\Delta(x)\) which satisfy conditions (1.24) and assumptions \((A_1)-(A_3)\). For such functions Theorem 1.1 gives new, easily verified criteria (inequalities (1.32) and (1.33)), which lead to the first complete classification of phaselocked solutions of (1.2). In Section 2 a specific example illustrates the two classes of behavior when (1.32) or (1.33) holds. Conclusions and statements of problems for future research are given in Section 3. The proof of Theorem 1.1 is in the Appendix.
2. Example. In this section we give a specific example which illustrates the predictions of Theorem 1.1. For this we consider a random variable derived from the Beta random variable, \( \beta(m, n) \). The range of definition of \( \beta(m, n) \) is

\[
0 \leq \beta(m, n) \leq 1, \quad 0 < m < \infty \text{ and } 0 < n < \infty,
\]

and its probability distribution function is defined by

\[
\rho(z; m, n) = \frac{\Gamma(m+n)}{\Gamma(n)\Gamma(m)} z^{m-1}(1 - z)^{n-1}, \quad 0 < z < 1.
\]

We restrict our attention to parameter regime \( D = \{m = 1, n \geq 1\} \cup \{n = 1, m \geq 1\} \), in which case the domain in (2.2) extends to the entire interval \( 0 \leq z \leq 1 \). When \((m, n) \in D\) we let \( Z(m, n) \) denote the random variable whose range is

\[
-1 \leq Z(m, n) \leq 1,
\]

and whose pdf \( f(z) \) is the symmetric extension of \( \rho(z; m, n) \) defined by

\[
f(z) = \begin{cases} 
\frac{\Gamma(m+n)}{2^{m+n}\Gamma(m)\Gamma(n)} (-z)^{m-1}(1 + z)^{n-1}, & -1 \leq z \leq 0 \\
\frac{\Gamma(m+n)}{2^{m+n}\Gamma(m)\Gamma(n)} z^{m-1}(1 - z)^{n-1}, & 0 \leq z \leq 1.
\end{cases}
\]

The \( \frac{1}{2} \) in (2.4) is a normalizing factor introduced so that \( \int_{-1}^{1} f(z)dz = 1 \).

It is easily verified that the mean and variance of \( Z(m, 1) \) and \( Z(1, n) \) satisfy

\[
\text{E}(Z(m, 1)) = 0 \quad \text{and} \quad \text{Var}(Z(m, 1)) = \frac{m}{m + 2}, \quad m \geq 1,
\]

\[
\text{E}(Z(1, n)) = 0 \quad \text{and} \quad \text{Var}(Z(1, n)) = \frac{n}{n + 2}, \quad n \geq 1.
\]

It follows from (2.4), (2.5) and (2.6) that \( f(z) \) satisfies requirements (1.24)-(1.15).

Goals. We consider parameter regimes \( D_1 = \{m = 1, n \geq 1\} \) and \( D_2 = \{n = 1, m \geq 1\} \) separately. In each of these regimes our goals are the following:

(a) Give the formula for \( f(z) \).
(b) Derive the formula for \( \Delta(x) \).
(c) Derive the formula for \( \gamma_0(m, n) \).
(d) Show how the hypotheses and conclusions of Theorem 1.1 are satisfied.

At the end of this section we give a numerical example (see Figures 2.1 and 2.2) which illustrates the application of Theorem 1.1 to parameter regime D.

(1.) Parameter Regime \( D_1 = \{m = 1, n \geq 1\} \)

(a) The formula for \( f(z) \). When \( m = 1 \) and \( n \geq 1 \), (2.4) reduces to

\[
f(z) = \begin{cases} 
\frac{1}{2}(1 + z)^{n-1}, & -1 \leq z \leq 0 \\
\frac{1}{2}(1 - z)^{n-1}, & 0 \leq z \leq 1.
\end{cases}
\]

When \( n = 1 \), (2.7) becomes the uniform distribution (Figure 2.1, Row 1)

\[
f(z) = \frac{1}{2}, \quad -1 \leq z \leq 1.
\]
(b) The formula for $\Delta(x)$. Substitute (2.7) into (1.19) and get

$$x = \begin{cases} \frac{1}{2}(1 + z)^n, & -1 \leq z \leq 0 \\ 1 - \frac{1}{2}(1 - z)^n, & 0 \leq z \leq 1. \end{cases}$$

Combining (2.9) with (1.20) gives

$$\Delta(x) = \begin{cases} (2x)^\frac{1}{n} - 1, & 0 \leq x < \frac{1}{2} \\ 1 - (2(1 - x))^\frac{1}{n}, & \frac{1}{2} \leq x \leq 1. \end{cases}$$

(c) The formula for $\gamma^*_*(1, n)$. It follows from (2.10) and (1.22) that

$$\gamma^*_*(1, n) = 2 \int_0^{1/2} \sqrt{1 - (2x)^\frac{1}{n} - 1} dx > 0, \quad n \geq 1,$$

We conclude from (2.11) that (Figure 2.2)

$$\gamma^*_*(1, 1) = \frac{\pi}{4}, \quad \gamma^*_*(1, n) \text{ increases as } n \geq 1 \text{ increases, and } \lim_{n \to \infty} \gamma^*_*(1, n) = 1.$$

(d) Proof that the conclusions of Part I of Theorem 1.1 hold.

It follows from (2.7), (2.10) and (2.11) that assumptions $(A_1)$-$(A_3)$ in Section 1 hold. Substituting (2.10) into equations (1.10) and (1.31) in Section 1 gives

$$C = 2 \int_0^{1/2} \sqrt{1 - \frac{\gamma^2}{C^2} \left(2x'\right)^\frac{1}{n} - 1} \, dx',$$

and

$$\frac{\partial H}{\partial C}(\gamma^*_*, \gamma^*_*) = 2 \gamma^*_* \int_0^{1/2} \frac{1 - 2 \left(2x\right)^\frac{1}{n} - 1}{\sqrt{1 - \left(2x\right)^\frac{1}{n} - 1}} \, dx.$$

The conclusions in Part I of Theorem 1.1 hold once we prove

$$\frac{\partial H}{\partial C}(\gamma^*_*, \gamma^*_*) \begin{cases} 0, & n = 1 \\ > 0, & n > 1 \end{cases}$$

In particular, it follows from (2.15) that criterion (1.32) in Theorem 1.1 is satisfied. Thus, properties (i)-(ii) in Theorem 1.1 hold: that is, when $m = 1$ and $n \geq 1$, $\gamma^*_*(1, n)$ is the phaselocking threshold, and (Figure 2.2)

(i) there is exactly one solution of (2.13) when $0 < \gamma \leq \gamma^*_*(1, n)$,

(ii) no solution of (2.13) when $\gamma^*_*(1, n) < \gamma \leq 1$.

It follows from (2.14) that (2.15) holds if we show that

$$\int_0^{1/2} \frac{1 - 2 \left(2x\right)^\frac{1}{n} - 1}{\sqrt{1 - \left(2x\right)^\frac{1}{n} - 1}} \, dx \begin{cases} 0, & n = 1 \\ > 0, & n > 1 \end{cases}$$
At the critical value \( n = 1 \) set \( 1 - 2x = \sin(\theta) \). Then the left side of (2.16) reduces to

\[
\int_0^{1/2} \frac{1 - 2(1 - 2x)^2}{\sqrt{1 - (1 - 2x)^2}} dx = \int_0^{\pi/2} (1 - 2\sin^2(\theta)) d\theta = 0.
\]

Next, a straightforward computation gives

\[
\frac{\partial}{\partial n} \int_0^{1/2} \frac{1 - 2 (2x)^{1/n} - 1}{\sqrt{1 - ((2x)^{1/n} - 1)^2}} dx
\]

(2.18) \[
= \frac{1}{n^2} \int_0^{1/2} \frac{((2x)^{1/n} - 1) \ln(2x) (2x)^{1/n} \left(3 - 2((2x)^{1/n} - 1)^2\right)}{(1 - ((2x)^{1/n} - 1)^2)^{3/2}} dx > 0, \quad n \geq 1.
\]

Properties (2.17) and (2.18) imply that (2.16) and (2.15) hold.

(2.) Parameter Regime \( D_2 = \{ n = 1, m > 1 \} \)

(a) The formula for \( f(z) \). When \( n = 1 \) and \( m > 1 \),

\[
f(z) = \begin{cases} 
\frac{m}{2} z^{m-1}, & 0 \leq z \leq 1 \\
\frac{m}{2} (-z)^{m-1}, & -1 \leq z \leq 0
\end{cases}
\]

(2.19)

In this case \( f(z) \) has a concave up “U-shape” (Figure 2.1, Row 2). Thus, \( f(z) \) is bimodal and the most probable value of \( Z \) is \( Z = \pm 1 \), which are equally probable. The study of U-shaped distributions dates back to the 1897 and 1927 classic papers by Pearson \[10\] and Rider \[11\]. These authors apply U-shaped distributions to widely diverse fields, including analysis of gaps in grades on mathematics exams, and the degrees of cloudiness at Breslau. Martens et al \[8\] make use of a combination of Cauchy-Lorentzian functions to model a bimodal frequency distribution in system (1.1). Bimodal distributions have recently been observed in ISI’s (inter spike intervals) of spike train outputs of bursting neurons in crickets \[7\].

(b) The formula for \( \Delta(x) \). Substituting (2.19) into (1.19) gives

\[
x = \begin{cases} 
\frac{1}{2} (1 - (-z)^m), & -1 \leq z \leq 0 \\
\frac{1}{2} (1 + z^m), & 0 \leq z \leq 1
\end{cases}
\]

(2.20)

Combining (2.20) with (1.20), we obtain

\[
\Delta(x) = \begin{cases} 
-(1 - 2z)^{-\frac{1}{m}}, & 0 \leq x < \frac{1}{2} \\
(2z - 1)^{-\frac{1}{m}}, & \frac{1}{2} \leq x \leq 1
\end{cases}
\]

(2.21)

(c) The formula for \( \gamma_*(m, 1) \). It follows from (2.21) and (1.22) that

\[
\gamma_*(m, 1) = 2 \int_0^{\frac{1}{2}} \sqrt{1 - (1 - 2x)^2/m} dx > 0, \quad 0 < m < \infty,
\]

(2.22)
We conclude from (2.22) that (Figure 2.2)

(2.23) \( \gamma_*(1, 1) = \frac{\pi}{4} \), \( \gamma_*(m, 1) \) decreases as \( m > 1 \) increases, and \( \lim_{m \to \infty} \gamma_*(m, 1) = 0 \).

(d) **Proof that the conclusions of Part II of Theorem 1.1 hold.**

It follows from (2.21) and (2.22) that assumptions (A_1)-(A_3) hold. Substituting (2.21) into equations (1.10) and (1.31) in Section 1 gives

(2.24) \[
C = 2 \int_0^{1/2} \sqrt{1 - \left(\frac{\gamma}{C}\right)^2 \left(1 - 2x\right)} \, dx',
\]

and

(2.25) \[
\frac{\partial H}{\partial C}(\gamma_*(m, 1), \gamma_*(m, 1)) = \frac{2}{\gamma_*(m, 1)} \int_0^{1/2} \frac{1 - 2(1 - 2x)^{2/m}}{\sqrt{1 - (1 - 2x)^{2/m}}} \, dx.
\]

The conclusions in **Part II** of Theorem 1.1 hold once we prove

(2.26) \[
\frac{\partial H}{\partial C}(\gamma_*(m, 1), \gamma_*(m, 1)) \left\{ \begin{array}{ll} 0, & m = 1 \\ < 0, & m > 1 \end{array} \right.
\]

In particular, when \( n = 1 \) and \( m > 1 \), it follows from (2.26) that criterion (1.33) in Theorem 1.1 is satisfied. Thus, properties (iii)-(iv)-(v)-(vi) in Theorem 1.1 hold: that is, the phaselocking threshold occurs at a value \( \gamma^*(m, 1) > \gamma_*(m, 1) \), and (Figure 2.2)

(iii) there is exactly one solution of (2.24) when \( 0 < \gamma < \gamma_*(m, 1) \),

(iv) exactly two solutions of (2.24) exist when \( \gamma_*(m, 1) \leq \gamma < \gamma^*(m, 1) \),

(v) exactly one solution of (2.24) exists when \( \gamma = \gamma^*(m, 1) \), and

(vi) no solution of (2.24) exists when \( \gamma^*(m, 1) < \gamma \leq 1 \).

It follows from (2.25) that property (2.26) holds if we show that

(2.27) \[
\int_0^{1/2} \frac{1 - 2(1 - 2x)^{2/m}}{\sqrt{1 - (1 - 2x)^{2/m}}} \, dx \left\{ \begin{array}{ll} 0, & m = 1 \\ < 0, & m > 1 \end{array} \right.
\]

At the critical value \( m = 1 \) we set \( 1 - 2x = \sin(\theta) \) and conclude from (2.28) that

(2.28) \[
\int_0^{1/2} \frac{1 - 2(1 - 2x)^2}{\sqrt{1 - (1 - 2x)^2}} \, dx = 0
\]

Next, a straightforward computation gives

\[
\frac{\partial}{\partial m} \int_0^{1/2} \frac{1 - 2(1 - 2x)^{2/m}}{\sqrt{1 - (1 - 2x)^{2/m}}} \, dx \]

(2.29) \[
= \frac{1}{m^2} \int_0^{1/2} \frac{\ln(1 - 2x)(1 - 2x)^{2/m}(3 - 2(1 - 2x)^{2/m})}{\left(1 - (1 - 2x)^{2/m}\right)^{3/2}} \, dx < 0, \quad m \geq 1.
\]

Properties (2.28) and (2.29) imply that (2.27) and (2.26) hold.

(iv) **Numerical Examples.**
Fig. 2.1. **Row 1** Graphs of \( f(z) \) defined in (2.7) when \((m, n) = (1, 2)\) (uni-modal distribution, left panel) and \((m, n) = (1, 1)\) (uniform distribution, right panel). **Row 2** Graph of bimodal U-shaped distribution \( f(z) \) defined in (2.19) when \((m, n) = (6, 1)\).

Figures 2.1 and 2.2 illustrate the predictions in Theorem 1.1 for specific values of \( m \) and \( n \). When \( m = 1 \) and \( n \geq 1 \) we substitute (2.7) into (1.18) and solve synchronization criterion

\[(2.30) \quad C = \int_0^1 \sqrt{1 - \frac{\gamma^2}{C^2} z^2 n(1 - z)^{n-1}} dz, \quad n \geq 1.\]

When \( n = 1 \) and \( m \geq 1 \) we substitute (2.21) into (1.18) and solve synchronization criterion

\[(2.31) \quad C = \int_0^1 \sqrt{1 - \frac{\gamma^2}{C^2} z^2 m z^{m-1}} dz, \quad m \geq 1.\]

The bifurcation diagram in Figure 2.2 shows three curves representing solutions of criterion (2.31) when \((m, n) = (6, 1)\), and criterion (2.30) when \((m, n) = (1, 2)\) and \((m, n) = (1, 1)\). The corresponding pdfs are shown in Figure 2.1.

3. **Conclusions.** In this paper we investigated existence of phase-locked solutions of (1.2) of the form

\[(3.1) \quad \theta(x, t) = \bar{\omega}t + \phi(x).\]

Ermentrout [4] conjectured that solutions of the form (3.1) exist and are stable when criterion (1.10) (or its equivalent formulation (1.18)) is satisfied. Our main theoretical advance (Theorem 1.1) is the derivation of two criteria ((1.32) and (1.33) in
Fig. 2.2. Bifurcation diagram showing curves of solutions of (2.30) bifurcating from the point $(\gamma_*(m,n), \gamma_*(m,n))$ on the line $C = \gamma$ (dashed) when $(m,n) = (1,1)$ and $(m,n) = (1,2)$, and a curve of bifurcating solutions of (2.31) when $(m,n) = (6,1)$. When $(m,n) = (1,1)$ or $(m,n) = (1,2)$ there is one solution for each $\gamma \in (0, \gamma_*(m,n)]$, as predicted by Part I of Theorem 1.1. When $(m,n) = (6,1)$ two solutions coexist when $\gamma \in \gamma_*(6,1), \gamma_*(6,1)) \approx (0.46, 0.57)$, as predicted by Part II of Theorem 1.1. At $(\gamma_*(1,1), \gamma_*(1,1)) = (\pi/4, \pi/4) \approx (0.78, 0.78)$ the bifurcation undergoes a transition from subcritical to supercritical as $(m,n)$ passes from parameter regime $D_1$ to parameter regime $D_2$.

Theorem 1.1) which guarantee either the existence of exactly one, or coexistence of exactly two, solutions of (1.10) over an interval of $\gamma$ values. Additionally, we obtain rigorous estimates for the value of phaselocking threshold. When criterion (1.33) in Theorem 1.1 holds we prove that the phaselocking threshold is greater than the value predicted in [4]. Physically important problems for future study are the following:

**Problem 1.** Prove the stability (or instability) of phase-locked solutions of (1.2).

**Problem 2.** The coexistence of two solutions of Ermentrout’s phase-locking criterion should have interesting implications for the behavior of the full model. More precisely, when two phase-locked solutions coexist, how do they affect the behavior of a solution $\theta(x,t)$ of (1.2) with general initial profile $\theta(x,0) = \theta_0(x), 0 \leq x \leq 1$?

4. Appendix. In this section we prove Theorem 1.1.

**Proof of Part (I).** In this part we assume that inequality (1.32) holds. That is,

\[
\frac{\partial H}{\partial C}(\gamma_*, \gamma_*) = \frac{1}{\gamma_*} \int_0^1 \frac{1 - 2\Delta^2(x)}{\sqrt{1 - \Delta^2}} dx \geq 0.
\]

Let $\gamma \in (0, \gamma_*)$ be fixed. To prove property (i) we need to analyze $H(\gamma, C)$ when $\gamma \leq C \leq 1$. Assumptions (A1)-(A3) and (1.23) imply that

\[
H(\gamma, 1) = 1 - \int_0^1 \sqrt{1 - \gamma^2 \Delta^2(x')} dx' > 0, \quad 0 < \gamma \leq 1.
\]
Property (1.29) shows that $H(\gamma, \gamma) < 0$. From this, (4.2), property (1.28) and the maximum principle it follows that there is a unique $C = C_1(\gamma) \in (\gamma, 1)$ such that $H(\gamma, C_1(\gamma)) = 0$. When $\gamma = \gamma_*$ we conclude from (1.29) and (4.1) that

$$H(\gamma_*, \gamma_*) = 0 \quad \text{and} \quad \frac{\partial H}{\partial C}(\gamma_*, \gamma_*) \geq 0. \quad (4.3)$$

It follows from (1.28) and (4.3) that

$$\frac{\partial H}{\partial C}(\gamma_*, C) > 0 \quad \text{and} \quad H(\gamma_*, \gamma_*) > 0 \quad \forall C \in (\gamma_*, 1). \quad (4.4)$$

Thus, $C = \gamma_*$ is the only solution of $H(\gamma_*, C) = 0$ when $\gamma_* \leq C \leq 1$, and the corresponding phase-locked solution of (1.1) is given by (1.30). This proves (i).

To prove (ii) we let $\gamma \in (\gamma_*, 1)$ be fixed. It follows from (1.23), (1.26), (1.29), (1.31) and (4.1) that

$$\frac{\partial H}{\partial C}(\gamma, \gamma) = \frac{1}{\gamma} \left( H(\gamma, \gamma) + \int_0^1 1 - 2\Delta^2(x) dx \right) > \frac{\gamma_*}{\gamma} \frac{\partial H}{\partial C}(\gamma_*, \gamma_*) \geq 0. \quad (4.5)$$

Combining (1.29) with (4.5) gives

$$H(\gamma, \gamma) > 0 \quad \text{and} \quad \frac{\partial H}{\partial C}(\gamma, \gamma) > 0, \quad \gamma_* < \gamma < 1. \quad (4.6)$$

Properties (1.28) and (4.6) imply that

$$H(\gamma, C) > 0 \quad \text{and} \quad \frac{\partial H}{\partial C}(\gamma, C) > 0 \quad \forall C \in (\gamma, 1). \quad (4.7)$$

From (4.7) it follows that there is no value $C \in [\gamma, 1]$ such that $H(\gamma, C) = 0$. This completes the proof of (ii).

**Proof of Part (II).** In this part we assume that inequality (1.33) holds. That is,

$$\frac{\partial H}{\partial C}(\gamma_*, \gamma_*) = \frac{1}{\gamma_*} \int_0^1 \frac{1 - 2\Delta^2(x)}{\sqrt{1 - \Delta^2}} dx < 0. \quad (4.8)$$

First, we let $0 < \gamma < \gamma_*$ be fixed and prove property (iii). In this case it follows exactly as in Part I that there is a unique $C = C_1(\gamma) \in (\gamma, 1)$ such that $H(\gamma, C_1(\gamma)) = 0$.

Next, we prove (iv) when $\gamma = \gamma_*$. It follows from (1.29) and (4.8) that

$$H(\gamma_*, \gamma_*) = 0 \quad \text{and} \quad \frac{\partial H}{\partial C}(\gamma_*, \gamma_*) < 0. \quad (4.9)$$

Thus, $C = \gamma_*$ is the first solution of $H(\gamma_*, C) = 0$, and the corresponding phase-locked solution of (1.1) is given by (1.30). Next, from (4.9) and continuity of $H(\gamma, C)$ and $\frac{\partial H}{\partial C}(\gamma, C)$ we conclude that there is an $\epsilon > 0$ such that

$$\frac{\partial H}{\partial C}(\gamma_*, C) < 0 \quad \text{and} \quad H(\gamma_*, C) < 0, \quad \gamma_* < C \leq \gamma_* + \epsilon. \quad (4.10)$$

It follows from (1.28), (4.2), (4.10) and the maximum principle that there is a unique $C_1(\gamma_*) \in (\gamma_*, 1)$ such that $H(\gamma_*, C_1(\gamma_*)) = 0$. Thus, when $C > \gamma_*$, $C = C_1(\gamma_*)$ is the
second and only solution of \(H(\gamma_*, C) = 0\). This proves (iv) when \(\gamma = \gamma_*\). To prove (iv) when \(\gamma > \gamma_*\) we use a continuation argument. Define the set

\[ S = \{ \hat{\gamma} \in (\gamma_*, 1) | \text{if } \gamma_* < \hat{\gamma} < \gamma \text{ then there exists } C_0(\gamma) \in (\gamma, 1) \text{ such that} \]

\[ (4.11) \quad H(\gamma, C) > 0 \text{ and } \frac{\partial H}{\partial C}(\gamma, C) < 0 \forall C \in (\gamma, C_0(\gamma)], \quad \text{and } H(\gamma, C_0(\gamma)) = 0 \]

We need to show that

\[ (4.12) \quad S \neq \emptyset, \text{ S is open and } \gamma^* = \sup S < 1. \]

Although the proof of (4.12) is somewhat technical, it is essential that we give complete details in order to make the completion of the proof of (iv) clear. The first step is to conclude from from assumption (A3) that \(H(\gamma, C)\) and \(\frac{\partial H}{\partial C}(\gamma, C)\) are uniformly continuous over the compact range

\[ (4.13) \quad \gamma_* \leq \gamma \leq 1 \text{ and } \gamma \leq C \leq 1. \]

From this fact, (1.29) and (4.10) we conclude that, if \(0 < \gamma - \gamma_* \ll 1\), then

\[ (4.14) \quad H(\gamma, \gamma) > 0, H(\gamma, \gamma + \epsilon) < 0, \quad \frac{\partial H}{\partial C}(\gamma, \gamma) < 0, \quad \text{and } \frac{\partial H}{\partial C}(\gamma, \gamma + \epsilon) < 0. \]

It follows from (4.14) and continuity of \(H(\gamma, C)\) that, if \(\gamma - \gamma_* > 0\) is sufficiently small, there is a \(C_0 \in (\gamma, \gamma + \epsilon)\) such that

\[ (4.15) \quad H(\gamma, C) > 0 \forall C \in [\gamma, C_0), \quad H(\gamma, C_0) = 0 \quad \text{and} \quad \frac{\partial H}{\partial C}(\gamma, C_0) \leq 0. \]

Suppose that \(\frac{\partial H}{\partial C}(\gamma, C_0) = 0\). It follows from (1.28) that \(\frac{\partial^2 H}{\partial C^2}(\gamma, C) > 0\) for all \(C \in [\gamma, 1]\). Thus, \(\frac{\partial H}{\partial C}(\gamma, C) > 0\) and \(H(\gamma, C) > 0\) for all \(C \in (C_0, \gamma + \epsilon]\), contradicting (4.14).

We conclude that \(\frac{\partial H}{\partial C}(\gamma, C_0) < 0\). The same reasoning shows that \(\frac{\partial H}{\partial C}(\gamma, C) < 0\) for all \(C \in [\gamma, C_0]\). This proves that \(\gamma \in S\) if \(\gamma - \gamma_* > 0\) is sufficiently small. Next, it follows from continuity of \(H(\gamma, C)\) and \(\frac{\partial H}{\partial C}(\gamma, C)\) over compact range (4.13) that \(S\) is an open set. It remains to prove that \(\gamma^* < 1\). First, it follows from (4.2) that \(H(1, 1) > 0\). This property and uniform continuity of \(H(\gamma, C)\) over compact range (4.13) imply that \(H(\gamma, C) > 0\) if \(1 - \gamma > 0\) is sufficiently small and \(\gamma \leq C \leq 1\). From this fact and the definition of \(S\) we conclude that \(\gamma^* = \sup S < 1\). This completes the proof of (4.12). We now complete the proof of (iv). It follows from the definition of \(S\) and properties (4.12) that

\[ (4.16) \quad S = (\gamma_*, \gamma^*). \]

Let \(\gamma \in (\gamma_*, \gamma^*)\) be fixed. We have shown that there is a value \(C_0(\gamma) \in (\gamma, 1)\) such that

\[ (4.17) \quad H(\gamma, C) > 0 \quad \text{and} \quad \frac{\partial H}{\partial C}(\gamma, C) < 0 \forall C \in (\gamma, C_0(\gamma)] \quad \text{and} \quad H(\gamma, C_0(\gamma)) = 0. \]

Thus, \(C = C_0(\gamma)\) is the first solution of \(H(\gamma, C) = 0\) when \(\gamma \in S = (\gamma_*, \gamma^*)\). Next, it follows from (4.17), and the continuity of \(H(\gamma, C)\) and \(\frac{\partial H}{\partial C}(\gamma, C)\) that there is a \(\delta > 0\) such that

\[ (4.18) \quad H(\gamma, C) < 0 \quad \text{and} \quad \frac{\partial H}{\partial C}(\gamma, C) < 0 \forall C \in (C_0(\gamma), C_0(\gamma) + \delta). \]
Recall from (4.2) that $H(\gamma, 1) > 0$. This, (4.18) and (1.28) imply that there is a unique $C_1(\gamma) \in (C_0(\gamma), 1)$ where $H(\gamma, C_1(\gamma)) = 0$. Thus, $C = C_1(\gamma)$ is the second and only solution of $H(\gamma, C) = 0$ when $\gamma \in S = (\gamma_*, \gamma^*)$. This completes the proof of (iv).

Next, we let $\gamma = \gamma^*$ and prove (v). It follows from (1.26) that

$$H(\gamma^*, \gamma^*) > 0. \tag{4.19}$$

Suppose that $H(\gamma^*, C) > 0$ for all $C \in [\gamma^*, 1]$. Then (4.19) and continuity of $H(\gamma, C)$ imply that

$$H(\gamma, C) > 0 \quad \forall C \in [\gamma, 1] \quad if \quad 0 < \gamma^* - \gamma < 1. \tag{4.20}$$

Thus, $\gamma \notin S$ when $0 < \gamma - \gamma^* < 1$. This implies that $\sup S > \gamma^*$, which contradicts (4.12). We conclude that there is a $C_* \in (\gamma^*, 1]$ such that

$$H(\gamma^*, C) > 0 \quad \forall C \in [\gamma^*, C_*], \quad H(\gamma^*, C_*) = 0 \quad and \quad \frac{\partial H}{\partial C}(\gamma^*, C_*) \leq 0. \tag{4.21}$$

We need to prove that

$$\frac{\partial H}{\partial C}(\gamma^*, C) < 0 \quad \forall C \in [\gamma^*, C_*]. \tag{4.22}$$

Suppose that $\frac{\partial H}{\partial C}(\gamma^*, C) \geq 0$ for some $\hat{C} \in (\gamma^*, C_*)$. Then (1.28) implies that $\frac{\partial H}{\partial C}(\gamma^*, C) > 0$ and $H(\gamma^*, C) > 0$ for all $C \in (\hat{C}, 1]$, contradicting (4.21). We conclude that property (4.22) holds. Next, (4.2) and continuity of $H(\gamma, C)$ imply that $C_* < 1$. If $\frac{\partial H}{\partial C}(\gamma^*, C_*) < 0$, then (4.21) and the definition of $S$ implies that $\gamma^* \in S$. Since $S$ is open, it follows that $\gamma \in S$ when $0 < \gamma - \gamma^* < 1$, contradicting the definition of $\gamma^*$. We conclude that $\frac{\partial H}{\partial C}(\gamma^*, C_*) = 0$. Combining this fact with convexity property (1.28) gives

$$\frac{\partial H}{\partial C}(\gamma^*, C) > 0 \quad and \quad H(\gamma, C) > 0 \quad \forall C \in (C_*, 1). \tag{4.23}$$

It follows from (4.21) and (4.23) that $C = C_*$ is the unique solution of $H(\gamma^*, C) = 0$ in the interval $(\gamma^*, 1)$. This completes the proof of (v).

Finally, we prove property (vi). First, we conclude from (1.27) that

$$\frac{\partial H}{\partial \gamma}(\gamma^*, C) > 0, \quad \gamma^* < \gamma < 1, \quad \gamma < C < 1. \tag{4.24}$$

Combining (4.21) and (4.23) with (4.24) gives

$$H(\gamma, C) > H(\gamma^*, C) \geq 0, \quad \gamma^* < \gamma < 1, \quad \gamma < C < 1. \tag{4.25}$$

It follows from (4.25) that the equation $H(\gamma, C) = 0$ has no solution when $\gamma^* < \gamma < 1$ and $\gamma \leq C < 1$. This proves (vi). The proof of Theorem 1.1 is now complete.

Acknowledgments. The author thanks Professor Stewart Anderson for pointing out U shaped distribution references.