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LETTER TO THE EDITOR

The mechanism of stochastic resonance

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Abstract. It is shown that a dynamical system subject to both periodic forcing and random perturbation may show a resonance (peak in the power spectrum) which is absent when either the forcing or the perturbation is absent.

The word resonance is usually applied in physics to cases in which a dynamical system, having periodic oscillations at some frequencies ω_i, when subject to a periodic forcing of frequencies near one of the ω_i, shows a marked response. The classical example is that of the forced harmonic oscillator.

In this Letter we investigate the possibility of resonance in dynamical systems which (in the absence of forcing) have a continuum power spectrum, or in other words behave stochastically. In this case the dynamical system has motion on all time scales.

We will show that for such systems there can also be a cooperative effect between the internal mechanism and the external periodic forcing. We shall call this effect stochastic resonance. We point out that this is a rather new phenomenon for stochastic dynamical systems and it is likely to have interesting applications.

To make clear our result we begin with an example in which a complete analytical theory can be developed. We describe the effect of stochastic resonance for the Langevin equation:

$$\mathrm{d}x = [x(a-x^2)]\mathrm{d}t + \varepsilon \mathrm{d}W \tag{1}$$

where W is a Wiener process. When $a<0$ the deterministic part of equation (1) has only one stable solution. At the bifurcation point a changes sign, and for $a>0$ there are two stable solutions $x_{1,2} = \pm\sqrt{a}$ and an unstable one $x=0$. We want to study the statistical properties of equation (1) subject to a small periodic forcing, i.e.

$$\mathrm{d}x = [x(a-x^2) + A\cos\Omega t]\,\mathrm{d}t + \varepsilon\,\mathrm{d}W. \tag{2}$$

We shall show that for $\varepsilon \in (\varepsilon_1, \varepsilon_2)$, where ε_1 and ε_2 will be estimated below, the system described by equation (2) has a large peak in the power spectrum corresponding to a nearly periodic behaviour of $x(t)$ with period $2\pi/\Omega$ and amplitude $2\sqrt{a}$.

First of all let us recall the most important statistical properties of equation (1) for $a>0$. Due to the random white noise, the solution of equation (1) jumps at random times between the two stable steady states. Let us call $\tau_1(y)$ and $\tau_2(y)$ the exit times

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from the basins of attraction of the points $x_1 = -\sqrt{a}$ and $x_2 = \sqrt{a}$ respectively, i.e.

$$\tau_1(y) = \inf(t\colon x(t) = 0 \text{ and } x(0) = y \in (-\infty, 0)),$$

$$\tau_2(y) = \inf(t\colon x(t) = 0 \text{ and } x(0) = y \in (+\infty, 0)).$$

Let us define $M'_n = \langle(\tau_i(y)^n)\rangle$ with $i = 1, 2$ and $M^i_0 = 1$. The $M^i_n(y)$ satisfies a differential equation (Gihman and Skorohod 1972):

$$\frac{1}{2}\varepsilon^2 \frac{\mathrm{d}^2}{\mathrm{d}y^2} M^i_n(y) - y(a - y^2)\frac{\mathrm{d}}{\mathrm{d}y} M^i_n(y) = -nM^i_{n-1}(y), \qquad i = 1, 2, \tag{3}$$

with boundary conditions $M^1_n(0) = 0$ and $M^2_n(0) = 0$. Using saddle point technique, we can estimate the solutions of equations (3). In particular, for $M^1_1(y)$ and $M^2_1(y)$ we obtain

$$M^1_1(y) \cong M^1_1(-\sqrt{a}) \cong (\pi/a\sqrt{2}) \exp(a^2/2\varepsilon^2) \tag{4}$$

and

$$M^2_1(y) \cong 2[M^1_1(-\sqrt{a})]^2 \cong (\pi^2/a^2) \exp(a^2/\varepsilon^2). \tag{5}$$

Note that because of the symmetry we have

$$M^2_1(\sqrt{a}) = M^1_1(-\sqrt{a}).$$

From equations (4) and (5) we see that the variance of the exit time is nearly equal to the mean exit time. It follows that no significant peak can be shown by the power spectrum of x.

Let us now discuss the property of equation (2). We are interested in the case where A is small compared with $a^{3/2}$. To understand the physical effect of the periodic forcing we being by discussing equation (2) for $t = 0$ and for $t = \Omega/\pi$. In other words, we discuss the two time-independent stochastic equations

$$\mathrm{d}x = [x(a - x^2) + A]\mathrm{d}t + \varepsilon\, \mathrm{d}W, \tag{6}$$

$$\mathrm{d}x = [x(a - x^2) - A]\mathrm{d}t + \varepsilon\, \mathrm{d}W. \tag{7}$$

Like equation (1), equations (6) and (7) have two stable fixed points and one unstable fixed point. However, there is no longer symmetry between the exit times from the two basins of attraction. Let us call x'_i the fixed points of equation (6) and x''_i the fixed points of equation (7). Using the same technique leading to estimates (4) and (5), we obtain

$$\mu(x'_1) \cong \frac{\pi}{a\sqrt{2}} \exp\left(\frac{a^2}{2\varepsilon^2}\left(1 + \frac{4A}{a^{3/2}}\right)\right), \tag{8}$$

$$\nu(x''_1) \cong \frac{\pi}{a\sqrt{2}} \exp\left(\frac{a^2}{2\varepsilon^2}\left(1 - \frac{4A}{a^{3/2}}\right)\right), \tag{9}$$

where $\mu(x'_1)$ and $\nu(x''_1)$ are the mean exit times from the basin of attraction to which x'_1 and x''_1 belong. We can now understand the qualitative behaviour of equation (2). Let us suppose we start at $t = 0$ with $x = x'_1$. As time passes, the probability to exit from the basin of attraction increases, and it reaches a maximum for $t = \pi/\Omega$. If we call τ the mean exit time to exit from the basin of attraction, it follows that

$$\nu(x''_1) < \tau < \mu(x'_1).$$

Now if

$$\mu(x_1') \gtrsim \pi/\Omega \qquad \text{and} \qquad \nu(x_1'') \ll \pi/\Omega \tag{10}$$

then $\tau \approx \pi/\Omega$, while the variance of the exit time is of order $\nu(x_1'')$. Therefore with probability near 1 the solution of equation (2) with initial condition $x = x_1'$ at $t = 0$ will jump to the point $x = x_2''$ at $t = \pi/\Omega$. In the same way, it is possible to see that the solution will spend a time about π/Ω in the new basin of attraction and at $t = 2\pi/\Omega$ will jump to the point $x = x_1'$. In this case $x(t)$ will jump between the two stable steady states nearly periodically in phase with the periodic forcing. According to equation (10), we see that in order to satisfy the inequalities, the variance of the noise has to be confined in the interval $(\varepsilon_1, \varepsilon_2)$ where ε_1 and ε_2 are given by

$$\varepsilon_1 = a\left(\frac{1-4A/a^{3/2}}{2\ln(2\sqrt{2}a/\Omega)}\right)^{1/2}, \qquad \varepsilon_2 = a\left(\frac{1+4A/a^{3/2}}{2\ln(2\sqrt{2}a/\Omega)}\right)^{1/2},$$

where for $\varepsilon = \varepsilon_1$, $\nu(x_1'') = \pi/\Omega$ and for $\varepsilon = \varepsilon_2$, $\mu(x_1') = \pi/\Omega$. In other words for a given value of A, small compared with $a^{3/2}$, the power structure of $x(t)$ shows a peak at the frequency when ε is confined between the values ε_1 and ε_2. This is what we call stochastic resonance. It has been recently applied by the authors to the study of climatic changes during the last 700 000 years (Benzi *et al* 1981), (see figure 1). We believe that this mechanism can also be important for those systems showing Hopf bifurcation and stochastic forcing, as recently described by Graham (1980).

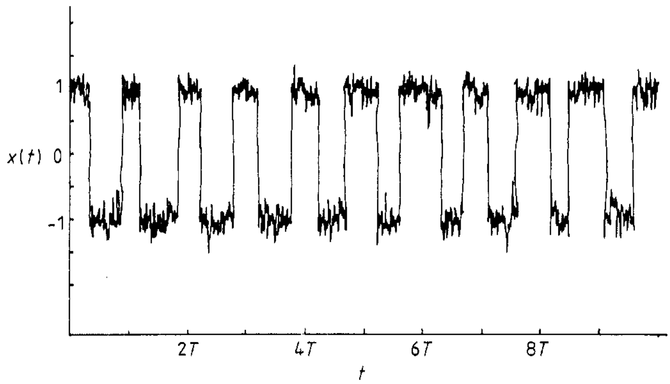

Figure 1. Numerical simulation of equation (2) with $a = 1$, $A = 0.12$, $\Omega = 2\pi/T = 10^{-3}$ and $\varepsilon = 0.25$. Note that using equation (10) we obtain $\varepsilon_1 = 0.18$ and $\varepsilon_2 = 0.31$. Therefore $\varepsilon \in (\varepsilon_1, \varepsilon_2)$.

We now discuss the stochastic resonance in a deterministic system whose solution is asymptotic to a strange attractor. It is well known that in this case the behaviour of the system is chaotic. The classical example is the Lorenz model described by

$$\dot{x} = \sigma(y - x), \qquad \dot{y} = rx - y - xz, \qquad \dot{z} = -bz + xy,$$

where $\sigma = 10$, $b = \frac{8}{3}$ (Lorenz 1963). For $r > r_c = 24.74$ the model shows chaotic behaviour. This model has been used by several authors as a prototype of the transition to chaos in deterministic dynamical systems, and it is a prototype for the transition to turbulence as well as laser dynamics (see Rabinovich (1978) for a detailed discussion).

We have studied the effect of a small periodic forcing $A \cos\Omega t$ on the Lorenz model, in the form

$$x = \sigma(y - x), \qquad y = rx - y - xz + A \cos\Omega t, \qquad z = -bz + xy, \qquad (11)$$

with $A = 30$ and $\Omega = 1$. We have computed the Fourier transform $\tilde{x}(\omega)$ of $x(t)$. For $r > r_c$ there is a marked peak in $|\tilde{x}(\omega)|^2$ for $\omega \sim \Omega$. In figure 2 we show $|\tilde{x}(\omega)|^2$ as a function of r. As can be seen, for $r > r_c$ we have a sudden transition to the stochastic resonance. A detailed analysis of this effect will be given in a forthcoming paper. We hope that the phenomenon of the stochastic resonance will be observed in experimental studies.

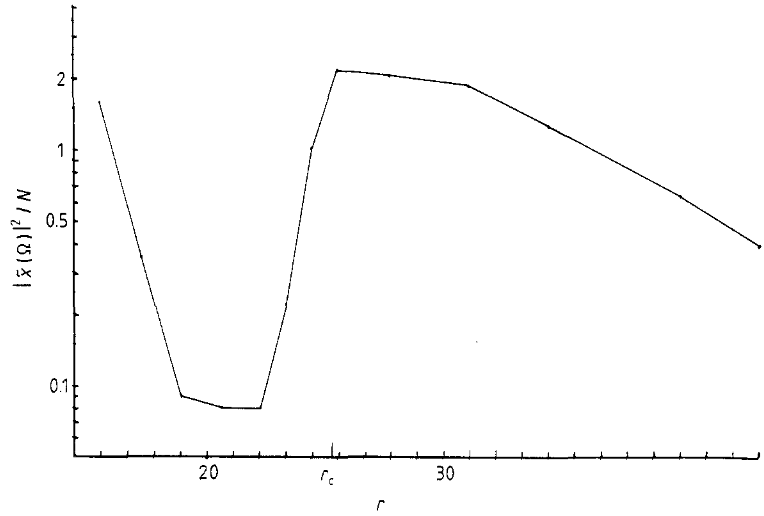

Figure 2. Plot of $|\tilde{x}(\omega)|^2/N$ against r for the Lorenz model equation (11). N is a normalising factor chosen arbitrarily. Note that for small value of r the periodic forcing is large enough to produce periodic oscillations between the two stable solutions. for increasing value of r, the effect of the periodic forcing is decreased as $r^{-3/2}$. The sudden jump of $|\tilde{x}(\omega)|^2$ near r_c is due, therefore, to the transition to the stochasticity of the Lorenz model.

The theory of the stochastic resonance for the Lorenz model cannot be developed analytically as in the case of equation (1). However, a qualitative discussion can be performed using recent investigations of Sutera (1980) and Zippelius and Lucke (1981). They studied the statistical properties of the Lorenz model subject to external white noise for both cases $r < r_c$ and $r > r_c$. First of all they found that the Lorenz model does not significantly change its statistical properties for $r > r_c$ when subject to external white noise. Moreover they both found striking similarity between the two statistical properties for $r < r_c$ and $r > r_c$. This observation suggests that the mechanism of stochastic resonance can be observed also for the Lorenz model stochastically perturbed by a white noise in the case $r < r_c$. Therefore the theory developed for equation (1) can be applied in this case, with the necessary complication in estimating the mean exit times from the two basins of attraction. This could serve as a guideline in developing a theory of stochastic resonance for the case $r > r_c$ with and without stochastic perturbations.

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