Properties of Linear Translation Invariant (LTI) Systems

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1 The continuous case

(a.) Definition of an LTI system. Let \( X(t) \) be a continuous time random variable (input).
The linear system is
\[
Y(t) = T(X(t))
\]
(1.1)

It is assumed that the system is time invariant, i.e.
\[
Y(t - t_0) = T(X(t - t_0)) = Y(t) \quad \forall t \text{ and } t_0.
\]
(1.2)

When the system (1.1) is linear and translationally invariant system it is referred to as an LTI system.
The key to everything is to analysis is write the input function \( X(t) \) in the integral equation form
\[
X(t) = \int_{-\infty}^{\infty} \delta(\lambda)X(t - \lambda)d\lambda.
\]
(1.3)

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Substitute (1.3) into (1.1) and get
\[ Y(t) = \int_{-\infty}^{\infty} h(\lambda)X(t - \lambda)d\lambda, \]  
(1.4)
where
\[ h(t) = T(\delta(t)) \]  
(1.5)
is the "impulse response function" for the linear system. Associated with \( h(t) \) is the "frequency response" of the system, i.e. the Fourier transform
\[ H(\omega) = \int_{-\infty}^{\infty} h(\lambda)e^{-i\omega\lambda}d\lambda \quad \forall \omega \in R. \]  
(1.6)
The autorcorrelation function for \( Y(t) \) is
\[ R_Y(t, s) = E[Y(t)Y(s)]. \]  
(1.7)
Substitution of (1.4) into (1.7) gives
\[ R_Y(t, s) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha)h(\beta)R_X(t - \alpha, s - \beta)d\alpha d\beta \]  
(1.8)
If \( X \) is a WSS continuous random variable then (1.8) becomes
\[ R_Y(t, s) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha)h(\beta)R_X(s - t + \alpha - \beta)d\alpha d\beta \]  
(1.9)
Let \( \tau = s - t + \alpha - \beta \). Then (1.9) further reduces to
\[ R_Y(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha)h(\beta)R_X(\tau + \alpha - \beta)d\alpha d\beta \]  
(1.10)
Remark This proves that \( Y(t) = T(X(t)) \) is a WSS r.v. if \( X(t) \) is a WSS r.v.
Next, the frequency power spectrum fpr \( Y \) is \( S_Y(\omega) \) is defined by
\[ S_Y(\omega) = \int_{-\infty}^{\infty} R_Y(\tau)e^{i\omega\tau}d\tau \quad \forall \omega \in R. \]  
(1.11)
Recall that
\[ S_Y(\omega) \] is real, \( S_Y(-\omega) = S_Y(\omega) \) and \( S_Y(\omega) \geq 0 \quad \forall \omega \in R. \)  
(1.12)
For example, \( R_Y(\tau) \) is real and even since \( R_X \) is real and even. The property \( S_Y \geq 0 \) is proved in power.pdf. Next, substitute (1.10) into (1.11) and obtain
\[ S_Y(\omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha)h(\beta)R_X(\tau + \alpha - \beta)d\alpha d\beta \exp(-i\omega\tau)d\tau \quad \forall \omega \in R. \]  
(1.13)
Let \( \eta = \tau + \alpha - \beta \). Then (1.13) separates into the triple product
\[ S_Y(\omega) = \int_{-\infty}^{\infty} h(\alpha)e^{i\omega\alpha}d\alpha \int_{-\infty}^{\infty} h(\beta)e^{-i\omega\beta}d\beta \int_{-\infty}^{\infty} R_X(\eta)e^{-i\omega\eta}d\eta \quad \forall \omega \in R. \]  
(1.14)
From this we obtain the important formula

\[ S_Y(\omega) = |H(\omega)|^2 S_X(\omega) \]  
(1.15)

The inverse of \( S_Y(\omega) \) is given by

\[ R_Y(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_Y(\omega) \exp(i\omega \tau) d\omega \]  
(1.16)

Substitution of (1.15) into (1.16) gives another important formula, namely

\[ R_Y(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 S_X(\omega) \exp(i\omega \tau) d\omega \]  
(1.17)

The quantity \( E(Y^2(t)) \) is defined to be the total power of process \( Y(t) \). Note that

\[ \text{Total Power} = E(Y^2(t)) = R_Y(t, t) = R_Y(t-t) = R_Y(0). \]  
(1.18)

From this and (1.16) it follows that

\[ \text{Total Power} = R_Y(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_Y(\omega) d\omega \]  
(1.19)

The area under the curve \( S_Y(\omega) \) is the total power of process \( Y(t) \). Finally, we combine (1.17) and (1.19) to obtain the famous formula

\[ \text{Total Power} = E(Y^2(t)) = R_Y(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 S_X(\omega) d\omega. \]  
(1.20)

(b.) Example 1. The generation of red noise from uniform white noise

Red noise \( Y(t) \) is obtained from white noise \( X(t) \) by solving the equation

\[ Y(t) = aY(t-1) + X(t) \quad \forall t \geq 1, \]  
(1.21)

where \( 0 < a < 1 \). Our goal is to derive the red noise autocorrelation function \( R_Y(\tau) \), and the associated power spectrum function \( S_Y(\omega) \). These functions are defined by

\[ S_Y(\omega) = |H(\omega)|^2 S_X(\omega) \quad \text{and} \quad R_Y(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_Y(\omega) \exp(i\omega \tau) d\omega. \]  
(1.22)

Because \( X(t) \) is uniform white noise, it must be the case that

\[ R_X(\tau) = \sigma^2 \delta(\tau) \quad \text{and} \quad S_X(\omega) = \sigma^2 \forall \omega. \]  
(1.23)

The function \( H(\omega) \) is defined by

\[ H(\omega) = \int_{-\infty}^{\infty} y(t) \exp(-i\omega t) dt, \]  
(1.24)

where \( y(t) \) is the solution of

\[ y(t) = ay(t-1) + \delta(t). \]  
(1.25)
Taking the Fourier transform of both sides of (1.25) gives
\[ \int_{-\infty}^{\infty} y(t) \exp(-i\omega t) dt = a \int_{-\infty}^{\infty} y(t-1) \exp(-i\omega t) dt + \int_{-\infty}^{\infty} \delta(t) \exp(-i\omega t) dt. \] (1.26)

Let \( \eta = t - 1 \). Then (1.26) becomes
\[ \int_{-\infty}^{\infty} y(t) \exp(-i\omega t) dt = a \exp(-i\omega) \int_{-\infty}^{\infty} y(\eta) \exp(-i\omega \eta) d\eta + \int_{-\infty}^{\infty} \delta(t) \exp(-i\omega t) dt. \] (1.27)

Substituting \( \int_{-\infty}^{\infty} y(t) \exp(-i\omega t) dt = H(\omega) \) and \( \int_{-\infty}^{\infty} \delta(t) \exp(-i\omega t) dt = 1 \) into (1.27) gives
\[ H(\omega) = \frac{1}{1 - a \exp(-i\omega)} \forall \omega. \] (1.28)

It follows from (1.22) and (1.28) that
\[ S_Y(\omega) = \frac{\sigma^2}{(1 - a \exp(-i\omega))(1 - a \exp(i\omega))} = \frac{\sigma^2}{1 - 2a \cos(\omega) + a^2}. \] (1.29)

Finally, combine (1.22) and (1.29), and use the Residue Theorem to obtain
\[ R_Y(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sigma^2 \exp(i\omega \tau)}{(1 - a \exp(-i\omega))(1 - a \exp(i\omega))} d\omega = \frac{\sigma^2}{1 - a^2} \delta(\tau) \forall \tau \geq 0. \] (1.30)

**Remarks.** For white noise recall that \( S_X(\omega) \) is constant since \( S_X(\omega) = \sigma^2 \\forall \omega \), and \( R_X(\tau) \) is the discontinuous step function given by \( R_X(\tau) = \sigma^2 \delta(\tau) \). For red noise, however, \( S_Y(\omega) \) is oscillatory in \( \omega \), and \( R_Y(\tau) \) is positive, continuous, and decreases as \( \tau \) increases from \( \tau = 0 \).

## 2 The discrete case

Let \( X(N) \) be a a discrete random variable. Then
\[ X(t) = X(N) \text{ if } t = N = \text{ an integer}, \quad \text{and} \quad X(t) = 0 \text{ if } t \neq \text{ an integer}. \] (2.1)

The linear system (1.1) takes the form
\[ Y(N) = T(X(N)) \forall N. \] (2.2)

As in the continuous case, it is assumed that the system is invariant, i.e.
\[ Y(N - N_0) = T(X(N - N_0)) = Y(N) \forall N \text{ and } N_0. \] (2.3)

Again, we write the input function \( X(N) \) in the integral equation form
\[ X(N) = \int_{-\infty}^{\infty} \delta(\lambda) X(N - \lambda) d\lambda. \] (2.4)

Because of (2.1) the integral in (2.4) becomes a Riemann sum, i.e.
\[ X(N) = \int_{-\infty}^{\infty} \delta(\lambda) X(N - \lambda) d\lambda = \sum_{j=-\infty}^{\infty} \delta(j) X(N - j). \] (2.5)
Next, substitute (2.4) into (2.2) and get
\[ Y(N) = \int_{-\infty}^{\infty} h(\lambda)X(n - \lambda)d\lambda, \] (2.6)
where
\[ h(\lambda) = T(\delta(\lambda)) \] (2.7)
As in (2.5), the integral in (2.6) also becomes the Riemann sum
\[ Y(N) = \sum_{j=-\infty}^{\infty} h(j)X(N - j), \] (2.8)
The "frequency response" of the system is the Fourier transform
\[ H(\Omega) = \int_{-\infty}^{\infty} h(\lambda)exp(-i\Omega\lambda)d\lambda \quad \forall \Omega \in R, \] (2.9)
which reduces to the Riemann sum
\[ H(\Omega) = \sum_{j=-\infty}^{\infty} h(j)exp(-i\Omega j) \quad \forall \Omega \in R. \] (2.10)
The autorcorrelation function for \( Y \) is
\[ R_Y(N, M) = E[Y(N)Y(M)]. \] (2.11)
Substitution of (2.8) into (2.11) gives
\[ R_Y(N, M) = \sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} h(j)h(l)R_X(N - j, M - l) \] (2.12)
If \( X \) is a WSS continuous random variable then (2.12) becomes
\[ R_Y(N, M) = \sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} h(j)h(l)R_X(M - N - j - l) \] (2.13)
Let \( M = N + k \). Then (2.13) further reduces to
\[ R_Y(N, N + k) = \sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} h(j)h(l)R_X(k + j - l) = R_Y(k), \] (2.14)
where \( R_Y(N, N + k) = R_Y(k) \) when \( k \) is an integer, and \( R_Y(k) = 0 \) when \( k \) is not an integer. The
frequency power spectrum
\[ S_Y(\Omega) = \int_{-\infty}^{\infty} R_Y(k)exp(-i\Omega k)dk \quad \forall \Omega \in R. \] (2.15)
reduces to the Riemann sum
\[ S_Y(\Omega) = \sum_{k=-\infty}^{\infty} R_Y(k)exp(-i\Omega k) \quad \forall \Omega \in R. \] (2.16)
Next, substitute (2.14) into (2.16) and get

\[ S_Y(\Omega) = \int_{-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} h(j)h(l)R_X(k+j-l)e^{i\Omega k}dk \ \forall \Omega \in R. \tag{2.17} \]

Let \( k + j - 1 = L \). Then (2.17) becomes the triple product

\[ S_Y(\Omega) = \sum_{j=-\infty}^{\infty} h(j)e^{i\Omega j} \sum_{l=-\infty}^{\infty} h(l)e^{i\Omega l} \int_{-\infty}^{\infty} R_X(L)e^{i\Omega L}dL \ \forall \Omega \in R. \tag{2.18} \]

Thus,

\[ S_Y(\Omega) = |H(\Omega)|^2S_X(\Omega) \ \forall \Omega \in R. \tag{2.19} \]

As in the continuous case,

\[ S_Y(\Omega) \text{ is real, } S_Y(-\Omega) = S_Y(\Omega) \text{ and } S_Y(-\Omega) \geq 0 \ \forall \Omega \in R. \tag{2.20} \]

The inverse of \( S_Y(\Omega) \) is given by

\[ R_Y(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_Y(\Omega)e^{i\Omega k}d\Omega \tag{2.21} \]

The area under the curve \( S_Y(\Omega) \) is the total power of process \( Y(t) \). That is,

\[ \text{Total Power} = E(Y^2(N)) = R_Y(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_Y(\Omega)d\Omega = |H(\Omega)|^2S_X(\Omega) \tag{2.22} \]