EXISTENCE AND EXACT MULTIPLICITY OF PHASELOCKED SOLUTIONS OF A KURAMOTO MODEL OF MUTUALLY COUPLED OSCILLATORS

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Abstract. We investigate existence and exact multiplicity of phase-locked solutions of an integro-differential equation derived from a Kuramoto system of coupled oscillators. Under general assumptions on the form of frequency distribution, we derive new, easily verified criteria which guarantee that either (i) exactly one solution exists, or (ii) exactly two solutions coexist over an entire interval of values of the key parameter $\gamma$. We illustrate our results with an example in which each of these possibilities occurs. Problems for future research are suggested.

Key words. integral equation, phase-locking, coupled system polynomial

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\[ \theta_j'(t) = \omega_j + \alpha \sum_{k=1}^{N} a_{jk}(\theta_k - \theta_j), \quad j = 1, \ldots, N, \]

where $\theta_j$ is the phase of the jth oscillator, $\omega_j$ is its natural frequency, and the all-to-all coupling function $a_{jk}(u)$ is periodic. This type of coupling has played an important role in simplifying the analysis of (1.1). For example, in the limit as $N \to \infty$, it is often possible to make use of the all-to-all coupling property to reduce (1.1) to a single integro-differential equation (see, for example, [2, 4, 6]). System (1.1) was used by Ermentrout and Kopell [7] to model frequency plateaus in the mammalian intestine. Cohen, Holmes, and Rand [5] used similar systems to model swimming in fish. In recent studies, various forms of (1.1) have been used to model the onset of synchronization in diverse settings [1, 4, 8, 10, 13]. In these studies the goal is to analyze a complex order parameter whose magnitude determines the degree of synchronization of the system. For this it is usually necessary to make further simplifying assumptions to complete the analysis. For example, in their model of chimera state synchronization, Abrams et al. [1] assume that the frequencies are not random, and all have the same value. In other settings, where frequencies are random [1, 4, 8, 10, 13], mathematical tractability of the order parameter function has required the use of Cauchy–Lorentzian distributions to model frequency distribution.

In this paper we follow Ermentrout’s alternative approach [6] to study synchronization in (1.1) which allows wider flexibility in the choice of frequency distribution. He assumes that each oscillator is coupled to every other one by $a_{jk}(u) = \frac{\sin(u)}{u}$ (the use of $\sin(u)$ as coupling function dates back to Adler [3] in 1946 and Turing [14] in 1952). Ermentrout derives a nonlinear equation ((1.12) and its equivalent formulation...
(1.20) below) which is a criterion for the existence of phase-locked solutions of (1.1). In order to accurately state our goals and main result (Theorem 1.1), we first need to give a brief description of the two step process in [6] to derive (1.12) and (1.20).

Step I. First, he assumes that $a_{jk}(u) = \frac{\sin(u)}{N}$ and that the frequencies of the oscillators are not randomly distributed, and takes the the continuum limit in (1.1) as $N \to \infty$ to obtain the integro-differential equation

$$\frac{\partial \theta}{\partial t} = \omega(x) + \alpha \int_0^1 \sin(\theta(x', t) - \theta(x, t)) \, dx',$$

where $t \geq 0$, $0 \leq x \leq 1$, and $\alpha > 0$. He assumes that the “frequency function” $\omega(x)$ is integrable on $[0, 1]$, that $\omega(x) = \bar{\omega} + \bar{\gamma} \Delta(x)$,

$$\sup_{0 \leq x \leq 1} |\Delta(x)| = 1, \quad \bar{\gamma} = \sup_{0 \leq x \leq 1} |\omega(x) - \bar{\omega}| > 0,$$

and

$$\bar{\omega} = \int_0^1 \omega(x) \, dx.$$

An important conclusion from (1.3) and (1.4) is that

$$\int_0^1 \Delta(x) = 0.$$

“Phase-locked” solutions of (1.2) have the form

$$\theta(x, t) = \bar{\omega}t + \phi(x).$$

Remark. If $\theta(x, t)$ satisfies (1.6) and $x \neq x'$ then we obtain the “phase-locked property”

$$\theta(x, t) - \theta(x', t) = \phi(x) - \phi(x')$$ independent of $t$.

Next, substituting (1.3) and (1.6) into (1.2) gives

$$\gamma \Delta(x) = \int_0^1 \sin(\phi(x) - \phi(x')) \, dx',$$

where $\gamma = \frac{\bar{\gamma}}{\alpha}$. A nontrivial phase-locked solution exists if there is a function $\phi(x)$ which satisfies (1.8) for some $\gamma > 0$. It follows from (1.8), and the restriction $\sup_{0 \leq x \leq 1} |\Delta(x)| = 1$, that $0 < \gamma \leq 1$. Next, observe that (1.8) is equivalent to

$$\gamma \Delta(x) = \sin(\phi(x)) \int_0^1 \cos(\phi(x')) \, dx' - \cos(\phi(x)) \int_0^1 \sin(\phi(x')) \, dx'.$$

Ermentrout claims, solely on the basis of numerical simulations, that stable phase-locked solutions exist when

$$\gamma \Delta(x) = C \sin(\phi(x)), -\frac{\pi}{2} \leq \phi(x) \leq \frac{\pi}{2}.$$
where
\begin{equation}
C = \int_0^1 \cos(\phi(x')) dx' = \int_0^1 \sqrt{1 - \sin^2(\phi(x'))} dx' \geq 0.
\end{equation}

Next, combine (1.9) with (1.5) and get \(\int_0^1 \sin(\phi(x')) dx' = 0\). Thus, (1.9) is consistent with (1.10)–(1.11). Finally, combining (1.10) with (1.11) gives the synchronization criterion
\begin{equation}
C = \int_0^1 \sqrt{1 - \frac{\gamma^2}{C^2} \Delta^2(x')} dx'.
\end{equation}
That is, a phase-locked solution exists if we find a value \(C\) which solves the fixed point problem (1.12). It follows from (1.12), and the restriction \(\sup_{0 \leq x \leq 1} |\Delta(x)| = 1\), that \(C \geq \gamma\). Also, we conclude from (1.12) that \(C \leq 1\). Thus, the range of \(\gamma\) and \(C\) is
\begin{equation}
0 < \gamma \leq 1 \quad \text{and} \quad \gamma \leq C \leq 1.
\end{equation}

If \((\gamma, C) = (\bar{\gamma}_\alpha, C)\) satisfies (1.12)–(1.13) then it follows from (1.3) and (1.10) that
\begin{equation}
\phi(x) = \sin^{-1} \left( \frac{\gamma \Delta(x)}{C} \right) = \sin^{-1} \left( \frac{\omega(x) - \bar{\omega}}{\alpha C} \right),
\end{equation}
and the corresponding phase-locked solution (1.6) of (1.2) is
\begin{equation}
\theta(x, t) = \bar{\omega} t + \sin^{-1} \left( \frac{\gamma \Delta(x)}{C} \right) = \bar{\omega} t + \sin^{-1} \left( \frac{\omega(x) - \bar{\omega}}{\alpha C} \right).
\end{equation}

Step II. Ermentrout [6] recasts (1.12) in terms of a probabilistic model. For this he assumes that the frequencies are randomly distributed and satisfy
\begin{equation}
\omega_j = \bar{\omega} + \bar{\gamma} Z_j, \quad j = 1, ..., N,
\end{equation}
where the \(Z_j\)'s are independent, identically distributed random variables, with range \(-1 \leq Z_j \leq 1\) and common PDF \(f(z)\). He assumes the following:
\begin{enumerate}
\item[(A0)] \(f(z) > 0\) is continuous and symmetric on \([-1, 1]\), and nonincreasing on \([0, 1]\).
\end{enumerate}
Thus, the mean and variance of \(Z_j\) satisfy
\begin{equation}
E(Z_j) = 0 \quad \text{and} \quad \text{Var}(Z_j) = \int_{-1}^1 z^2 f(z) dz \leq 1, \quad j = 1, ..., N.
\end{equation}

It follows from (1.16) and (1.17) that
\begin{equation}
E(\omega_j) = \bar{\omega} \quad \text{and} \quad \text{Var}(\omega_j) = \bar{\gamma}^2 \int_{-1}^1 z^2 f(z) dz \leq \bar{\gamma}^2, \quad j = 1, ..., N.
\end{equation}
Next, Ermentrout considers the discrete equation
\begin{equation}
C = \frac{1}{N} \sum_{j=1}^N \sqrt{1 - \frac{\gamma^2}{C^2} Z_j^2},
\end{equation}
and uses the law of large numbers to conclude that, in the limit \( N \to \infty \), \( C \) satisfies

\[
C = \int_0^1 \sqrt{1 - \frac{\gamma^2}{C^2} z^2} f(z) dz.
\]

(1.20)

Assumption (A_0) implies that (1.20) is equivalent to (1.12). To see this, let

\[
x = \int_{-1}^z f(\tau) d\tau, \quad -1 \leq z \leq 1.
\]

(1.21)

It follows from (1.21) and (A_0) that \( x(z) \) is an increasing function of \( z \in [-1, 1] \). Therefore \( x(z) \) is invertible and there is a unique function \( \Delta(x) \) such that

\[
z = \Delta(x(z)), \quad -1 \leq z \leq 1.
\]

(1.22)

Substituting (1.21)–(1.22) into (1.20) gives

\[
C = \int_0^1 \sqrt{1 - \frac{\gamma^2}{C^2} z^2} f(z) dz = \int_0^1 \sqrt{1 - \frac{\gamma^2}{C^2} \Delta^2(x')} dx'.
\]

(1.23)

It follows from (1.23) that synchronization criteria (1.12) and (1.20) are equivalent.

When \( f(z) \) satisfies (A_0), Ermentrout shows that there is at least one solution of (1.12) and (1.20) for each \( \gamma \in (0, \gamma^*) \), where

\[
\gamma^* = \int_{-1}^1 \sqrt{1 - z^2} f(z) dz = \int_0^1 \sqrt{1 - \Delta^2(x')} dx'.
\]

(1.24)

He claims, based on numerical simulation, that solutions of (1.12), or its equivalent formulation (1.20), are stable. An important step towards proving stability is to determine whether solutions of (1.12) and (1.20) are unique. However, Ermentrout does not investigate uniqueness. Instead, his main focus is to show that, if \( f(z) \) satisfies (A_0), then \( \gamma^* \) is the “phase-locking threshold,” the largest \( \gamma \in (0, 1) \) where a phase-locked solution can exist.

Our purpose is to give the first complete proof of existence, and exact multiplicity, of solutions of phase-locking criteria (1.12) and (1.20). Our assumptions on \( f(z) \) and \( \Delta(x) \) (see (A_1)–(A_3) below) include (A_0), but are significantly wider ranging than (A_0). In particular, we remove the restriction that \( f(z) \) is nonincreasing on \([0, 1]\) (see (1.26) below). We derive two new general criteria (inequalities (1.34) and (1.35) in Theorem 1.1) which guarantee that either (I) exactly one solution of (1.12) and (1.20) exists, or (II) exactly two solutions coexist, over an entire interval of values of the parameter \( \gamma \). Our most novel result is the prediction that two solutions coexist when criterion (1.35) holds. When two solutions coexist we prove that (i) \( f(z) \) lies outside the range of Ermentrout’s assumption (A_0), and (ii) the phase-locking threshold is greater than the value \( \gamma^* \) predicted in [6] (see the example in section 2).

We focus on analyzing the function

\[
H(\gamma, C) = C - \int_0^1 \sqrt{1 - \frac{\gamma^2}{C^2} \Delta^2(x')} dx'.
\]

(1.25)

Phase-locked solutions exist when \( H(\gamma, C) = 0 \).

We make the following assumptions:
(A1). \( \Delta(x) \) is derived from a PDF \( f(z) \) which is nonnegative and satisfies

\[
(1.26) \quad f(z) \text{ is continuous, symmetric, and } \int_{-1}^{z} f(\eta) \, d\eta \text{ is increasing on } [-1,1].
\]

Remarks. The PDF \( f(z) \) can oscillate multiple times on \([-1,1]\). Also, the property "\( \int_{-1}^{z} f(\eta) \, d\eta \) is increasing on \([-1,1]\)" is an essential property that allows for the inversion of \( x = \int_{-1}^{z} f(\eta) \, d\eta \).

(A2). \( \Delta(x) \) is continuous and increasing on \([0,1]\), satisfies (1.3)–(1.5), and

\[
(1.27) \quad \gamma_* = \int_{0}^{1} (1 - \Delta^2(x'))^{1/2} \, dx' > 0.
\]

(A3). \( H(\gamma, C), \frac{\partial H}{\partial C}(\gamma, C), \frac{\partial H}{\partial \gamma}(\gamma, C), \) and \( \frac{\partial^2 H}{\partial C^2}(\gamma, C) \) are continuous over parameter range (1.13), where they satisfy

\[
(1.28) \quad \frac{\partial H}{\partial C}(\gamma, C) = 1 - \frac{\gamma^2}{C^3} \int_{0}^{1} \frac{\Delta^2(x')}{(1 - \frac{\gamma^2}{C^2} \Delta^2(x'))^{1/2}} \, dx',
\]

\[
(1.29) \quad \frac{\partial H}{\partial \gamma}(\gamma, C) = \frac{\gamma}{C^2} \int_{0}^{1} \frac{\Delta^2(x')}{(1 - \frac{\gamma^2}{C^2} \Delta^2(x'))^{1/2}} \, dx' > 0,
\]

\[
(1.30) \quad \frac{\partial^2 H}{\partial C^2}(\gamma, C) = \frac{3\gamma^2}{C^4} \int_{0}^{1} \frac{\Delta^2(x)}{(1 - \frac{\gamma^2}{C^2} \Delta^2(x'))^{1/2}} \, dx + \frac{\gamma^4}{C^6} \int_{0}^{1} \frac{\Delta^4(x)}{(1 - \frac{\gamma^2}{C^2} \Delta^2(x'))^{3/2}} \, dx > 0.
\]

Before stating our main result we make three important observations. First, it follows from (1.3), (1.25), and assumptions \((A1)\)–\((A3)\) that

\[
(1.31) \quad H(\gamma, \gamma) = \gamma - \gamma_* \begin{cases} < 0, & 0 < \gamma < \gamma_*, \\ = 0, & \gamma = \gamma_*, \\ > 0, & \gamma_* < \gamma \leq 1, \end{cases}
\]

where \( \gamma_* \) is defined in (1.27). Property (1.31) plays an important role in the proof of our main result. Second, because \( H(\gamma_*, \gamma_*) = 0 \), we conclude that a phase-locked solution \( \theta_*(x, t) \) of (1.2) exists when \( (\gamma, C) = (\gamma_*, \gamma_*) \), and is given by

\[
(1.32) \quad \theta_*(x,t) = \bar{\omega}t + \phi_*(x), \quad \phi_*(x) = \sin^{-1} \left( \frac{\omega(x) - \bar{\omega}}{\alpha \gamma_*} \right).
\]

This property also plays an important role in the proof of our main result.

Third, combine (1.27) with (1.28) and get (see the appendix for complete details)

\[
(1.33) \quad \frac{\partial H}{\partial C}(\gamma_*, \gamma_*) = \frac{1}{\gamma_*} \int_{0}^{1} \frac{1 - 2\Delta^2(x')}{\sqrt{1 - \Delta^2(x')}} \, dx'.
\]

**Theorem 1.1.** Let \( f(z), \Delta(x), \gamma_* \), and \( H(\gamma, C) \) satisfy (1.25), (1.26), and assumptions \((A1)\)–\((A3)\).

(I) Suppose that

\[
(1.34) \quad \frac{\partial H}{\partial C}(\gamma_*, \gamma_*) = \frac{1}{\gamma_*} \int_{0}^{1} \frac{1 - 2\Delta^2(x')}{\sqrt{1 - \Delta^2(x')}} \, dx' \geq 0.
\]
Then $\gamma_*$ is the phase-locking threshold, and the following properties hold:

(i) if $0 < \gamma \leq \gamma_*$ then there is exactly one solution of (1.12);
(ii) if $\gamma_* < \gamma \leq 1$ then there is no solution of (1.12).
(II) Suppose that

$$
(1.35) \quad \frac{\partial H}{\partial \gamma}(\gamma_*, \gamma) = \frac{1}{\gamma_*} \int_0^1 \frac{1 - 2\Delta'(x')}{\sqrt{1 - \Delta^2(x')}} dx' < 0.
$$

Then there is a value $\gamma^* > \gamma_*$ such that $\gamma^*$ is the phase-locking threshold, and the following properties hold:

(iii) if $0 < \gamma < \gamma_*^*$ then there is exactly one solution of (1.12);
(iv) if $\gamma_*^* \leq \gamma < \gamma^*$ then there are exactly two solutions of (1.12);
(v) if $\gamma = \gamma^*$ then there is exactly one solution of (1.12);
(vi) if $\gamma^* < \gamma \leq 1$ then there is no solution of (1.12).

Remark. The precise definition of $\gamma^*$ is given in (A.20) in the appendix where we prove Theorem 1.1.

There is a wide diversity of functions $f(z)$ and $\Delta(x)$ which satisfy conditions (1.26) and assumptions (A1)–(A3). For such functions Theorem 1.1 gives new, easily verified criteria (inequalities (1.34) and (1.35)), which lead to the first complete classification of phase-locked solutions of (1.2). In section 2 a specific example illustrates the two classes of behavior when (1.34) or (1.35) holds. Conclusions and statements of problems for future research are given in section 3. The derivation of (1.33) and the proof of Theorem 1.1 are in the appendix.

2. Example. In this section we give an example which illustrates the predictions of Theorem 1.1. For this we examine a random variable derived from the beta random variable, $\beta(m, n)$. The range of definition of $\beta(m, n)$ is

$$
(2.1) \quad 0 \leq \beta(m, n) \leq 1, \quad 0 < m < \infty \text{ and } 0 < n < \infty,
$$

and its PDF is defined by

$$
(2.2) \quad \rho(z; m, n) = \frac{\Gamma(m + n)}{\Gamma(n)\Gamma(m)} z^{m-1}(1 - z)^{n-1}, \quad 0 < z < 1.
$$

We restrict our attention to parameter regime

$$
D = \{(m, n) \mid m = 1, n \geq 1\} \cup \{(m, n) \mid n = 1, m \geq 1\},
$$

in which case the domain in (2.2) extends to the entire interval $0 \leq z \leq 1$. When $(m, n) \in D$ we let $Z(m, n)$ denote the random variable whose range is

$$
(2.3) \quad -1 \leq Z(m, n) \leq 1,
$$

and whose pdf $f(z)$ is the symmetric extension of $\rho(z; m, n)$ defined by

$$
(2.4) \quad f(z) = \begin{cases} 
\frac{\Gamma(m+n)}{\Gamma(n)\Gamma(m)} (-z)^{m-1}(1 + z)^{n-1}, & -1 \leq z \leq 0, \\
\frac{\Gamma(m+n)}{\Gamma(n)\Gamma(m)} z^{m-1}(1 - z)^{n-1}, & 0 \leq z \leq 1.
\end{cases}
$$

The $\frac{1}{2}$ in (2.4) is a normalizing factor introduced so that $\int_{-1}^1 f(z)dz = 1$. 

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It is easily verified that the mean and variance of $Z(m, 1)$ and $Z(1, n)$ satisfy

(2.5) $E(Z(m, 1)) = 0$ and $\text{Var}(Z(m, 1)) = \frac{m}{m + 2}, \ m \geq 1,$

(2.6) $E(Z(1, n)) = 0$ and $\text{Var}(Z(1, n)) = \frac{n}{n + 2}, \ n \geq 1.$

It follows from (2.4), (2.5), and (2.6) that $f(z)$ satisfies requirements (1.17) and (1.26).

**Goals.** We investigate existence and uniqueness of phase-locked solutions in parameter regimes

$$D_1 = \{(m, n) \mid m = 1, n \geq 1\} \text{ and } D_2 = \{(m, n) \mid n = 1, m \geq 1\}.$$

In each of these regimes our goals are the following.

(a) Give the formula for $f(z)$.
(b) Derive the formula for $\Delta(x)$.
(c) Derive the formula for $\gamma^*(m, n)$.
(d) Show how the hypotheses and conclusions of Theorem 1.1 are satisfied.

At the end of this section we give a specific example (see Figures 1 and 2) which illustrates the application of Theorem 1.1 to parameter regime D.

1. **Parameter regime** $D_1 = \{(m, n) \mid m = 1, n \geq 1\}$.

   (a) **The formula for** $f(z)$. When $m = 1$ and $n \geq 1$, (2.4) reduces to

   $$f(z) = \begin{cases} \frac{1}{2}(1 + z)^{n - 1}, & -1 \leq z \leq 0 \\ \frac{1}{2}(1 - z)^{n - 1}, & 0 < z \leq 1. \end{cases}$$

   When $n = 1$, the pdf $f(z)$ defined in (2.7) is the uniform distribution (Figure 1, row 1, right panel)

   $$f(z) = \frac{1}{2}, \ -1 \leq z \leq 1.$$  

   In this case every value of $Z \in [-1, 1]$ is equally probable. When $n > 1$, $f(z)$ is unimodal and the most probable value of $Z$ is $Z = 0$ (Figure 1, row 1, left panel).

   (b) **The formula for** $\Delta(x)$. Substitute (2.7) into (1.21) and get

   $$x = \begin{cases} \frac{1}{2}(1 + z)^{n}, & -1 \leq z \leq 0, \\ 1 - \frac{1}{2}(1 - z)^{n}, & 0 < z \leq 1. \end{cases}$$

   Combining (2.9) with (1.22) gives

   $$\Delta(x) = \begin{cases} (2x)^{\frac{n}{2}} - 1, & 0 \leq x \leq \frac{1}{2}, \\ 1 - (2(1 - x))^{\frac{n}{2}}, & \frac{1}{2} < x \leq 1. \end{cases}$$

   (c) **The formula for** $\gamma^*(1, n)$. It follows from (2.10) and (1.24) that

   $$\gamma^*(1, n) = 2 \int_0^{\frac{1}{2}} \sqrt{1 - ((2x)^{\frac{n}{2}} - 1)^2} \, dx > 0, \ n \geq 1.$$
We conclude from (2.11) that (Figure 2)
\[
\gamma_\ast(1, 1) = \frac{\pi}{4}, \quad \gamma_\ast(1, n) \text{ increases as } n \geq 1 \text{ increases, and}
\]
(2.12)\[\lim_{n \to \infty} \gamma_\ast(1, n) = 1.\]

(d) Proof that the conclusions of part I of Theorem 1.1 hold. It follows from (2.7), (2.10), and (2.11) that assumptions (A_1)–(A_3) in section 1 hold. Substituting (2.10) into (1.12) and (1.33) in section 1 gives
\[
(2.13)\[C = 2 \int_0^{1/2} \sqrt{1 - \frac{\gamma_\ast^2}{C^2} \left((2x')^{\frac{1}{2a}} - 1\right)^2} \, dx'.\]

and
\[
(2.14)\[\frac{\partial H}{\partial C}(\gamma_\ast(1, n), \gamma_\ast(1, n)) = \frac{2}{\gamma_\ast} \int_0^{1/2} \frac{1 - 2((2x')^{\frac{1}{2a}} - 1)^2}{\sqrt{1 - ((2x')^{\frac{1}{2a}} - 1)^2}} \, dx.\]

The conclusions in part I of Theorem 1.1 hold once we prove
\[
(2.15)\[\frac{\partial H}{\partial C}(\gamma_\ast, \gamma_\ast) \left\{\begin{array}{ll}
= 0, & n = 1, \\
> 0, & n > 1.
\end{array}\right.\]
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Fig. 2. Two curves in \((C, \gamma)\) parameter space representing families of phase-locked solutions of (2.30) when \((m, n) = (1, 1)\) and \((m, n) = (1, 2)\), and a third curve representing the family of solutions of (2.31) when \((m, n) = (6, 1)\). Each curve begins at \((C, \gamma) = (1, 0)\) and ends at the point \((\gamma_*(m, n), \gamma_*(m, n))\) on the dashed line \(C = \gamma\). When \((m, n) = (1, 1)\) or \((m, n) = (1, 2)\) the corresponding curves have a “monotonic decreasing” shape, which implies that there is exactly one solution for each \(\gamma \in (0, \gamma_*(m, n))\), as predicted by part I of Theorem 1.1. However, when \((m, n) = (6, 1)\) the curve has a “hysteresis” shape which implies that exactly two solutions coexist when \(\gamma \in [\gamma_*(6, 1), \gamma^*(6, 1)] \approx (.46, .57)\), as predicted by Part II of Theorem 1.1. At \((\gamma_*(1, 1), \gamma_*(1, 1)) = (\pi/4, \pi/4) \approx (.78, .78)\) the shape of the curves undergo a transition from monotonic decreasing shape to hysteresis shape as \((m, n)\) passes from parameter regime \(D_1\) to parameter regime \(D_2\).

In particular, it follows from (2.15) that criterion (1.34) in Theorem 1.1 is satisfied. Thus, properties (i)–(ii) in Theorem 1.1 hold; that is, when \(m = 1\) and \(n \geq 1\), \(\gamma_*(1, n)\) is the phase-locking threshold, and (see Figure 2)

(i) there is exactly one solution of (2.13) when \(0 < \gamma \leq \gamma_*(1, 1)\),

(ii) there is no solution of (2.13) when \(\gamma_*(1, n) < \gamma \leq 1\).

It follows from (2.14) that (2.15) holds if we show that

\[
\int_0^{1/2} \frac{1 - 2((2x)^{1/2} - 1)^2}{\sqrt{1 - ((2x)^{1/2} - 1)^2}} \, dx \begin{cases} = 0, & n = 1, \\ > 0, & n > 1. \end{cases}
\]

At the critical value \(n = 1\), set \(1 - 2x = \sin(\theta)\). Then the left side of (2.16) reduces to

\[
\int_0^{1/2} \frac{1 - 2(1 - 2x)^2}{\sqrt{1 - (1 - 2x)^2}} \, dx = \int_0^{\pi/2} (1 - 2 \sin^2(\theta)) \, d\theta = 0.
\]
Next, a straightforward computation gives

\[
\frac{\partial}{\partial n} \int_{0}^{1} \frac{1 - 2((2x)^{1/n} - 1)^2}{\sqrt{1 - ((2x)^{1/n} - 1)^2}} dx
\]

\[= \frac{1}{n^2} \int_{0}^{1} (2x)^{1/n} \ln((2x)^{1/n} - 1) \frac{1}{(1 - ((2x)^{1/n} - 1)^2)^{3/2}} dx > 0, \quad n \geq 1.\]  

(2.18)

Properties (2.17) and (2.19) imply that (2.16) and (2.15) hold.

2. **Parameter regime**

\(D_2 = \{(m,n) \mid n = 1, m > 1\}\).

(a) The formula for \(f(z)\). When \(n = 1\) and \(m > 1\),

\[
(2.19) \quad f(z) = \begin{cases} 
\frac{m}{2} (-z)^{m-1}, & -1 \leq z \leq 0, \\
\frac{m}{2} z^{m-1}, & 0 \leq z \leq 1.
\end{cases}
\]

In this case \(f(z)\) has a concave up “U-shape” (Figure 1, Row 2). Thus, \(f(z)\) is bimodal and the most probable values of \(Z\) are \(Z = \pm 1\), which are equally probable. The study of U-shaped distributions dates back to the 1897 and 1927 classic papers by Pearson [11] and Rider [12]. These authors apply U-shaped distributions to widely diverse fields, including analysis of gaps in grades on mathematics exams, and the degrees of cloudiness at Breslau. Martens et al. [10] make use of a combination of Cauchy–Lorentzian functions to model a bimodal frequency distribution in system (1.1). Bimodal distributions have recently been observed in ISI’s (interspike intervals) of spike train outputs of bursting neurons in crickets [9].

(b) The formula for \(\Delta(x)\). Substituting (2.19) into (1.21) gives

\[
(2.20) \quad x = \begin{cases} 
\frac{1}{2} (1 - (-z)^m), & -1 \leq z \leq 0, \\
\frac{1}{2} (1 + z^m), & 0 < z \leq 1.
\end{cases}
\]

Combining (2.20) with (1.22), we obtain

\[
(2.21) \quad \Delta(x) = \begin{cases} 
-(1 - 2x)^{\frac{1}{m}}, & 0 \leq x \leq \frac{1}{2}, \\
(2x - 1)^{\frac{1}{m}}, & \frac{1}{2} < x \leq 1.
\end{cases}
\]

(c) The formula for \(\gamma_*(m,1)\). It follows from (2.21) and (1.24) that

\[
(2.22) \quad \gamma_*(m,1) = 2 \int_{0}^{\frac{1}{2}} \left(1 - 2x\right)^{\frac{1}{2m}} dx > 0, \quad 0 < m < \infty.
\]

We conclude from (2.22) that (Figure 2)

\[
\gamma_*(1,1) = \frac{\pi}{4}, \quad \gamma_*(m,1) \text{ decreases as } m > 1 \text{ increases, and}
\]

\[
(2.23) \quad \lim_{m \to \infty} \gamma_*(m,1) = 0.
\]
Specific Examples. Figures 1 and 2 illustrate the predictions in Theorem 1.1 for specific values of 

(2.30) 

\[ C = \int_0^1 \sqrt{1 - \frac{\gamma^2}{C^2}} z^2 n(1 - z)^{n-1} dz, \quad n \geq 1. \]
When \( n = 1 \) and \( m \geq 1 \) we substitute (2.21) into (1.20) and solve

\[
C = \int_0^1 \sqrt{1 - \gamma^2 C^2 z^2 m z^{-1}} \, dz, \quad m \geq 1.
\]

Figure 2 shows three curves in the \((C, \gamma)\) plane representing phase-locked solutions. The monotonic curves labeled \((m, n) = (1, 2)\) and \((m, n) = (1, 1)\) occur when criterion (2.30) holds. The nonmonotonic hysteresis curve labeled \((m, n) = (6, 1)\) occurs when criterion (2.31) holds. The corresponding PDFs are shown in Figure 1. Our proof of Theorem 1 predicts these shapes whenever criterion (2.30) or criterion (2.31) is satisfied.

3. Conclusions. In this paper we investigated the existence of phase-locked solutions of (1.2) of the form

\[
\theta(x, t) = \bar{\omega} t + \phi(x).
\]

Ermentrout [6] conjectured that solutions of the form (3.1) exist and are stable when criterion (1.12) (or its equivalent formulation (1.20)) is satisfied. Our main theoretical advance (Theorem 1.1) is the derivation of two criteria ((1.34) and (1.35) in Theorem 1.1) which guarantee either the existence of exactly one, or coexistence of exactly two, solutions of (1.12) over an interval of \( \gamma \) values. Additionally, we obtain rigorous estimates for the value of the phase-locking threshold. When criterion (1.35) in Theorem 1.1 holds we prove that the phase-locking threshold is greater than the value predicted in [6]. Physically important problems for future study are the following:

Problem 1. Prove the stability (or instability) of phase-locked solutions of (1.2). When two phase-locked solutions coexist it is possible, and even expected, that one is unstable.

Problem 2. The coexistence of two phase-locked solutions should have interesting implications. More precisely, when two phase-locked solutions coexist, how do they affect the behavior of a solution \( \theta(x, t) \) of (1.2) with general initial profile \( \theta(x, 0) = \theta_0(x), \ 0 \leq x \leq 1? \)

Appendix A. This section has two parts:
(I) We show how (1.33) in section 1 is derived from (1.27) and (1.28).
(II) We prove Theorem 1.1.

Part (I). We need to show how (1.33), i.e.,

\[
\frac{\partial H}{\partial C}(\gamma, \gamma) = \frac{1}{\gamma} \int_0^1 \frac{1 - 2 \Delta^2(x')}{\sqrt{1 - \Delta^2(x')}} \, dx',
\]

is derived from (1.27) and (1.28), i.e., from

\[
\gamma = \int_0^1 (1 - \Delta^2(x'))^{1/2} \, dx' > 0
\]

and

\[
\frac{\partial H}{\partial C}(\gamma, C) = 1 - \frac{\gamma^2}{C^2} \int_0^1 \frac{\Delta^2(x')}{(1 - \gamma^2 \Delta^2(x'))^{1/2}} \, dx'.
\]

First, recall from (1.25) that

\[
H(\gamma, C) = C - \int_0^1 \sqrt{1 - \frac{\gamma^2}{C^2} \Delta^2(x')} \, dx'.
\]
Taking the partial derivative of both sides of (A.4) with respect to \( C \) gives (A.3). In (A.3) set \( C = \gamma = \gamma_* \), and get

\[
\frac{\partial H}{\partial C}(\gamma_*, \gamma_*) = 1 - \frac{1}{\gamma_*} \int_0^1 \frac{\Delta^2(x')}{(1 - \Delta^2(x'))^{1/2}} dx'.
\]

Next, in (A.5) factor out the term \( \frac{1}{\gamma_*} \) to the left, and obtain

\[
\frac{\partial H}{\partial C}(\gamma_*, \gamma_*) = \frac{1}{\gamma_*} \left( \gamma_* - \int_0^1 \frac{\Delta^2(x')}{(1 - \Delta^2(x'))^{1/2}} dx' \right).
\]

In (A.6) replace the \( \gamma_* \) inside the parenthesis with (A.2), and obtain

\[
\frac{\partial H}{\partial C}(\gamma_*, \gamma_*) = \frac{1}{\gamma_*} \left( \gamma_* - \int_0^1 \frac{\Delta^2(x')}{(1 - \Delta^2(x'))^{1/2}} dx' \right).
\]

Combining the two integrals in (A.7) gives

\[
\frac{\partial H}{\partial C}(\gamma_*, \gamma_*) = \frac{1}{\gamma_*} \left( \int_0^1 \left( 1 - \Delta^2(x') \right)^{1/2} dx' - \int_0^1 \frac{\Delta^2(x')}{(1 - \Delta^2(x'))^{1/2}} dx' \right).
\]

Finally, combine the two components of the integrand in (A.8) into a single fraction, and obtain (A.1).

**Part (II).** In this part we prove Theorem 1.1. First, it is assumed that inequality (1.34) holds. That is,

\[
\frac{\partial H}{\partial C}(\gamma_*, \gamma_*) = \frac{1}{\gamma_*} \gamma_* - \int_0^1 \frac{1 - 2\Delta^2(x')}{\sqrt{1 - \Delta^2(x')}} dx' \geq 0.
\]

Let \( \gamma \in (0, \gamma_*) \) be fixed. To prove property (i) in Theorem 1.1 we need to analyze \( H(\gamma, C) \) when \( \gamma \leq C \leq 1 \). Assumptions \( (A_1)-(A_3) \) and (1.25) imply that

\[
H(\gamma, 1) = 1 - \int_0^1 \frac{1 - 2\Delta^2(x')}{\sqrt{1 - \Delta^2(x')}} dx' > 0, \quad 0 < \gamma \leq 1.
\]

Property (1.31) shows that \( H(\gamma, \gamma) < 0 \). This, (A.10), and convexity property (1.30) imply that there is a unique \( C = C_1(\gamma) \in (\gamma, 1) \) such that \( H(\gamma, C_1(\gamma)) = 0 \).

Next, when \( \gamma = \gamma_* \) we conclude from (1.31) and (A.9) that

\[
H(\gamma_*, \gamma_*) = 0 \quad \text{and} \quad \frac{\partial H}{\partial C}(\gamma_*, \gamma_*) \geq 0.
\]

It follows from (1.30) and (A.11) that

\[
\frac{\partial H}{\partial C}(\gamma_*, C) > 0 \quad \text{and} \quad H(\gamma_*, C) > 0 \forall C \in (\gamma_*, 1].
\]

Thus, \( C = \gamma_* \) is the only solution of \( H(\gamma_*, C) = 0 \) when \( \gamma_* \leq C \leq 1 \), and the corresponding phase-locked solution of (1.1) is given by (1.32). This proves (i).

To prove property (ii) in Theorem 1.1 we let \( \gamma \in (\gamma_*, 1) \) be fixed. It follows from (1.25), (1.28), (1.31), (1.33), and (A.9) that

\[
\frac{\partial H}{\partial C}(\gamma, \gamma) = \frac{1}{\gamma} \left( H(\gamma, \gamma) + \int_0^1 \frac{1 - 2\Delta^2(x')}{\sqrt{1 - \Delta^2(x')}} dx' \right) > \frac{\gamma_*}{\gamma} \frac{\partial H}{\partial C}(\gamma_*, \gamma) \geq 0.
\]
Combining (1.31) with (A.13) gives

(A.14) \[ H(\gamma, \gamma) > 0 \text{ and } \frac{\partial H}{\partial C}(\gamma, \gamma) > 0, \quad \gamma_* < \gamma < 1. \]

Properties (1.30) and (A.14) imply that

(A.15) \[ H(\gamma, C) > 0 \text{ and } \frac{\partial H}{\partial C}(\gamma, C) > 0 \forall C \in (\gamma, 1]. \]

From (A.15) it follows that there is no value \( C \in [\gamma, 1] \) such that \( H(\gamma, C) = 0 \). This completes the proof of (ii).

Next, we prove properties (iii)–(vi) in Theorem 1.1 under the assumption that

(A.16) \[ \frac{\partial H}{\partial C}(\gamma_*, \gamma_*) = \frac{1}{\gamma_*} \int_0^1 \frac{1 - 2\Delta^2(x')}{\sqrt{1 - \Delta^2(x')}} dx' < 0. \]

First, we let \( 0 < \gamma < \gamma_* \) be fixed and prove property (iii). In this case it follows exactly as above that there is a unique \( C = C(\gamma) \in (\gamma, 1) \) such that \( H(\gamma, C(\gamma)) = 0 \).

Next, we prove (iv) when \( \gamma = \gamma_* \). It follows from (1.31) and (A.16) that

(A.17) \[ H(\gamma_*, \gamma_*) = 0 \quad \text{and} \quad \frac{\partial H}{\partial C}(\gamma_*, \gamma_*) < 0. \]

Thus, \( C = \gamma_* \) is the first solution of \( H(\gamma_*, C) = 0 \), and the corresponding phase-locked solution of (1.1) is given by (1.32). Next, from (A.17) and continuity of \( H(\gamma_*, C) \) and \( \frac{\partial H}{\partial C}(\gamma_*, C) \) we conclude that there is an \( \epsilon > 0 \) such that

(A.18) \[ \frac{\partial H}{\partial C}(\gamma_*, C) < 0 \quad \text{and} \quad H(\gamma_*, C) < 0, \quad \gamma_* < C < \gamma_* + \epsilon. \]

It follows from (A.10), (A.18), and convexity property (1.30) that there is a unique \( C_1(\gamma_*) \in (\gamma_*, 1) \) such that \( H(\gamma_*, C_1(\gamma_*)) = 0 \). This proves property (iv) of Theorem 1.1 when \( \gamma = \gamma_* \). To prove (iv) when \( \gamma > \gamma_* \) we use a continuation argument. Define the set

\[ S = \{ \hat{\gamma} \in (\gamma_*, 1) \mid \text{if } \gamma_* < \hat{\gamma} < \gamma \text{ then there exists } C_0(\gamma) \in (\gamma, 1) \text{ such that} \]

(A.19) \[ \frac{\partial H}{\partial C}(\gamma, C) < 0 \forall C \in (\gamma, C_0(\gamma)], \quad \text{and} \quad H(\gamma, C_0(\gamma)) = 0 \} \]

We need to show that

(A.20) \[ S \neq \emptyset, \quad S \text{ is open}, \quad \text{and} \gamma^* = \sup S < 1. \]

Although the proof of (A.20) is somewhat technical, it is essential that we give complete details in order to make the completion of the proof of (iv) clear. The first step is to conclude from assumption (A.3) that \( H(\gamma, C) \) and \( \frac{\partial H}{\partial C}(\gamma, C) \) are uniformly continuous over the compact range

(A.21) \[ \gamma_* \leq \gamma \leq 1 \quad \text{and} \quad \gamma \leq C \leq 1. \]

From this fact, (1.31), and (A.18) we conclude that, if \( 0 < \gamma - \gamma_* << 1 \), then

(A.22) \[ H(\gamma, \gamma) > 0, \quad H(\gamma, \gamma + \epsilon) < 0, \quad \frac{\partial H}{\partial C}(\gamma, \gamma) < 0, \quad \text{and} \quad \frac{\partial H}{\partial C}(\gamma, \gamma + \epsilon) < 0. \]
It follows from (A.22) and continuity of $H(\gamma, C)$ and $\frac{\partial H}{\partial C}(\gamma, C)$ that, if $\gamma - \gamma_*>0$ is sufficiently small, there is a $C_0 = C_0(\gamma) \in (\gamma, \gamma + \epsilon)$ such that

(A.23) \quad H(\gamma, C) > 0 \quad \forall C \in [\gamma, C_0), \quad H(\gamma, C_0) = 0, \quad \text{and} \quad \frac{\partial H}{\partial C}(\gamma, C_0) \leq 0.

Suppose that $\frac{\partial H}{\partial C}(\gamma, C_0) = 0$. It follows from (1.30) that $\frac{\partial^2 H}{\partial C^2}(\gamma, C) > 0$ for all $C \in [\gamma, 1]$. Thus, $\frac{\partial H}{\partial C}(\gamma, C) > 0$ and $H(\gamma, C) > 0$ for all $C \in (C_0, \gamma + \epsilon]$, contradicting (A.22). We conclude that $\frac{\partial H}{\partial C}(\gamma, C_0) < 0$. The same reasoning shows that $\frac{\partial H}{\partial C}(\gamma, C) < 0$ for all $C \in [\gamma, C_0]$. This proves that $\gamma \in S$ if $\gamma - \gamma_*>0$ is sufficiently small. Next, it follows from continuity of $H(\gamma, C)$ and $\frac{\partial H}{\partial C}(\gamma, C)$ over compact range (A.21) that $S$ is an open set. It remains to prove that $\gamma^* < 1$. First, it follows from (A.10) that $H(1, 1) > 0$. This property and uniform continuity of $H(\gamma, C)$ over compact range (A.21) imply that $H(\gamma, C) > 0$ if $1 - \gamma > 0$ is sufficiently small and $\gamma \leq C \leq 1$. From this fact and the definition of $S$ we conclude that $\gamma^* = \sup S < 1$. This completes the proof of (A.20). We now complete the proof of property (iv). It follows from the definition of $S$ and properties (A.20) that

(A.24) \quad S = (\gamma_*, \gamma^*).

Let $\gamma \in (\gamma_*, \gamma^*)$ be fixed. We have shown that there is a $C_0(\gamma) \in (\gamma, 1)$ such that

(A.25) \quad H(\gamma, C) > 0 \quad \text{and} \quad \frac{\partial H}{\partial C}(\gamma, C) < 0 \quad \forall C \in [\gamma, C_0(\gamma)] \quad \text{and} \quad H(\gamma, C_0(\gamma)) = 0.

Thus, $C = C_0(\gamma)$ is the first solution of $H(\gamma, C) = 0$ when $\gamma \in S = (\gamma_*, \gamma^*)$. Next, it follows from (A.25), and the continuity of $H(\gamma, C)$ and $\frac{\partial H}{\partial C}(\gamma, C)$ that there is a $\delta > 0$ such that

(A.26) \quad H(\gamma, C) < 0 \quad \text{and} \quad \frac{\partial H}{\partial C}(\gamma, C) < 0 \quad \forall C \in (C_0(\gamma), C_0(\gamma) + \delta).

Recall from (A.10) that $H(\gamma, 1) > 0$. This, (A.26), and convexity property (1.30) imply that there is a unique $C_1(\gamma) \in (C_0(\gamma), 1)$ where $H(\gamma, C_1(\gamma)) = 0$. This completes the proof of property (iv).

Next, we let $\gamma = \gamma^*$ and prove (v). It follows from (1.28) that

(A.27) \quad H(\gamma^*, \gamma^*) > 0.

Suppose that $H(\gamma^*, C) > 0$ for all $C \in [\gamma^*, 1]$. Then convexity property (1.30) and continuity of $H(\gamma, C)$ imply that

(A.28) \quad H(\gamma, C) > 0 \quad \forall C \in [\gamma, 1] \quad \text{if} \quad 0 < \gamma^* - \gamma < \epsilon.

Thus, $\gamma^* \notin S$ when $0 < \gamma - \gamma^* < \epsilon$. This implies that $\sup S > \gamma^*$, which contradicts (A.20). We conclude that there is a $C_* \in (\gamma^*, 1]$ such that

(A.29) \quad H(\gamma^*, C) > 0 \quad \forall C \in [\gamma^*, C_*), \quad H(\gamma^*, C_*) = 0, \quad \text{and} \quad \frac{\partial H}{\partial C}(\gamma^*, C_*) \leq 0.

We claim that

(A.30) \quad \frac{\partial H}{\partial C}(\gamma^*, C) < 0 \quad \forall C \in [\gamma^*, C_*).

Suppose that $\frac{\partial H}{\partial C}(\gamma^*, C) \geq 0$ for some $C \in (\gamma^*, C_*).$ Then (1.30) implies that $\frac{\partial H}{\partial C}(\gamma^*, C) > 0$ and $H(\gamma^*, C) > 0$ for all $C \in (\tilde{C}, 1)$, contradicting (A.29). We conclude that
property \((A.30)\) holds. Next, \((A.10)\) and continuity of \(H(\gamma, C)\) imply that \(C_* < 1.\)
If \(\frac{\partial H}{\partial C}(\gamma^*, C_*) < 0,\) then \((A.29)\) and the definition of \(S\) imply that \(\gamma^* \in S.\) Since \(S\) is open,
it follows that \(\gamma \in S\) when \(0 < \gamma - \gamma^* << 1,\) contradicting the definition of \(\gamma^*.\) We conclude that \(\frac{\partial H}{\partial C}(\gamma^*, C_*) = 0.\) Combining this fact with convexity property
\((1.30)\) gives
\[
(A.31) \quad \frac{\partial H}{\partial C}(\gamma^*, C) > 0 \quad \text{and} \quad H(\gamma, C) > 0 \quad \forall C \in (C_*, 1).
\]
It follows from \((A.29), (A.30),\) and \((A.31)\) that \(C = C_*\) is the unique solution of \(H(\gamma^*, C) = 0\) in the interval \((\gamma^*, 1).\) This completes the proof of property (v).

Finally, we prove property (vi). First, we conclude from \((1.29)\) that
\[
(A.32) \quad \frac{\partial H}{\partial \gamma}(\gamma, C) > 0, \quad \gamma^* < \gamma < 1, \quad \gamma \leq C < 1.
\]
Combining \((A.29)\) and \((A.31)\) with \((A.32)\) gives
\[
(A.33) \quad H(\gamma, C) > H(\gamma^*, C) \geq 0, \quad \gamma^* < \gamma < 1, \quad \gamma \leq C < 1.
\]
It follows from \((A.33)\) that the equation \(H(\gamma, C) = 0\) has no solution when \(\gamma^* < \gamma < 1\) and \(\gamma \leq C < 1.\) This proves property (vi). The proof of Theorem 1.1 is now complete.

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**References**