Holditch's Theorem

A fresh look at a long-forgotten theorem.

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Hamnet Holditch, president of Cajus College in Cambridge during the middle part of the last century, discovered a remarkable property of a curve traced out by a point on a chord of fixed length that slides around with both endpoints on a convex curve. His proof, published in 1858 [4], makes a number of unstated assumptions and utilizes some notions unfamiliar to the modern reader. Yet the elementary part of the theorem is accessible to a good high school student, and a much more general result can be proved as a rather elementary application of line integrals in the plane.

In this paper we shall discuss the original presentation of Holditch and point out several limitations of his approach. We shall then describe a much more general theorem which is considerably easier to prove, and conclude by giving some applications to kinematics and to a polygonal version of the theorem in which the curve traced out by the point on the chord of fixed length is apparently of a sort quite different from that originally envisioned by Hamnet Holditch.

The material in this article was the subject of lectures by the author in March 1979 at the University of Washington and Simon Fraser University.

Holditch's Theorem, the Classical Case.

We begin with a problem, elementary enough for a good high school student. In a circle c of radius r, a chord is divided into parts of length a and b by point D. As the chord makes a complete sweep around the circle, an inner circle, the locus of D, is traced out. (See FIGURE 1(a), where the dotted curve is the locus.) Find the area of the ring-shaped region between the circle c and the locus of D.

\[ \text{FIGURE 1} \]
For the solution, we refer to the notation in Figure 1(b). Since in the circle, two intersecting chords are divided at $D$ into segments of lengths $a$, $b$, and $r+z$, $r-z$, respectively, it follows (see [3]) that $a \cdot b = (r+z) \cdot (r-z) = r^2 - z^2$. Thus $\pi ab = \pi r^2 - \pi z^2$, which shows that the area sought is $\pi ab$.

It is interesting to note that the answer to this problem is in terms of the segments $a$ and $b$ of the divided chord, and is wholly independent of $r$, the radius of the circle. Holditch observed that when the outer circle was replaced by a more general curve, the result remained correct.

**Holditch's Theorem** (his own formulation [4]). If a chord of a closed curve, of constant length $a+b$, be divided into two parts of lengths $a$, $b$, respectively, the difference between the areas of the closed curve, and of the locus of the dividing point, will be $\pi ab$.

**Figure 2**

**Figure 2(a)** shows a closed convex curve $c$ and some of its chords of a fixed length. In his proof, Holditch made the assumption that for the given curve $c$, all of the chords of fixed length $a+b$ would be tangent to another curve $e$ called the envelope of the family of chords. He let $Q$ denote the point where the chord $AB$ touches this envelope, and he let $r$ denote the length of the segment $AQ$ (which can vary with the position of the chord). Denoting $P$ as the dividing point of the chord, so $|AP| = a$, $|PB| = b$, (see Figure 2(b)), he then claims that the areas swept out by the segments $AQ$, $BQ$, and $QP$ during one revolution of the chord can be written as

$$I_1 = \frac{1}{2} \int_0^{2\pi} r^2 d\theta, \quad I_2 = \frac{1}{2} \int_0^{2\pi} (a+b-r)^2 d\theta, \quad I_3 = \frac{1}{2} \int_0^{2\pi} (a-r)^2 d\theta$$

respectively. The first two integrals are equal since $AQ$ and $BQ$ sweep out the same annular region between $e$ and $c$. The equation $I_1 = I_2$ and a short computation show that $\int_0^{2\pi} r d\theta = \pi (a+b)$. From this, it is readily shown that $I_2 - I_3$ (i.e., the area of the annular region swept out by $AP$ and shaded in Figure 2 (c)) is $\pi ab$. This is the desired result.

**Comments on Holditch's Version. A Modern Reformulation.**

The careful reader has already noticed several implicit assumptions made by Holditch in his formulation and proof. First, generalizing from the case of a circle, Holditch probably took $c$ to be a convex curve, but he does not mention this assumption.
Next, we note that the integral representations in his proof are known to be valid for the area swept out by a vector based at a fixed point and making one revolution counterclockwise, the length of the vector varying continuously. Although this property can be extended to a vector based at a variable point that goes around a convex curve, the vector always having the direction of the positive half-tangent of the curve, Holditch takes this extension for granted without comment. (Perhaps in Holditch’s day these results were well known.)

We have already noted that Holditch assumed the chords of length \( a + b \) are all tangent to the envelope \( e \); actually, Holditch does not use the term “envelope.” He says instead, in the language of that day: “Let \( Q \) be the point in which the chord intersects its consecutive position.” This terminology is probably influenced by Newton.

A rather serious omission from the statement of the theorem is his failure to mention that some upper bound for the length of the chord \( AB \) is needed. To see this need, let a chord \( AB \) of length \( \sqrt{2} \) glide along a rectangle \( KLNM \) whose sides have the lengths 1 and 2 (see Figure 3), and study the locus of the midpoint \( D \) of the chord. The chord is too long to slide along the entire rectangle in such a way that each point on the rectangle is hit by both \( A \) and \( B \). For example, if the chord \( AB \) begins in the position shown in Figure 3(a) and moves counterclockwise, both \( B \) and \( A \) travel along the upper edge of the rectangle until \( B \) reaches \( L \), then \( B \) travels down the left side of the rectangle to \( M \) (Figure 3(b)). Now, as endpoint \( B \) travels toward \( N \), endpoint \( A \) travels back to \( K \) (retracing its path) along the upper edge of the rectangle (Figure 3(c)). The chord can continue its travel around the rectangle, \( A \) sliding down the right edge to \( N \) and \( B \) retracing its path back to \( M \). Continuing until all possible paths have been traced, the locus of \( D \) consists of four line segments (two of them coincide) and four circle arcs. (The locus of \( D \) is shown by the dotted curve in Figure 3(d).) The point set between the rectangle and the locus has the area \( 1/2 + \pi/2 \). This differs from the area given by the conclusion of Holditch’s theorem (i.e., from the area \( \pi/2 \)). It is possible that Holditch assumed that the chord \( AB \) could travel in a simple way (always moving in the same direction) around the curve \( c \), with both endpoints \( A \) and \( B \) eventually hitting every point on \( c \).

![Figure 3](image-url)
The need for an upper bound for the chord length is also seen in Figure 4, which depicts the ellipse \( \frac{x^2}{2} + y^2 = 1 \) and the locus of \( D \) in the cases \( a = b = 0.7, \ a = b = 0.95, \ a = b = 1.1 \). In this situation, the locus of \( D \) is a simple closed curve for \( a = b < 1 \), and it is eight-shaped for \( a = b \geq 1 \); the conclusion in Holditch's theorem holds only for \( a = b \leq 1 \). This example shows that Holditch probably assumed that the locus of \( D \) was a simple closed curve.

Finally, Holditch overlooks the possibility that the envelope of the chords may not be of the simple type illustrated in Figure 1(a). (The assumption that this envelope is a convex curve is implicit in his derivation of the integrals in his proof.) Consider, for example, the chords of length 1 in the unit square (see Figure 5). Their envelope consists of several branches; one branch is the arc in the first quadrant of the astroid \( x^{2/3} + y^{2/3} = 1 \); other branches are obtained by successive rotations of this arc 90° around the point \((1/2, 1/2)\). Branches corresponding to chords parallel to a coordinate axis do not exist. Holditch's proof can be construed as covering the situation in Figure 5, provided a suitable (but not standard) definition of envelope is considered. The dotted circle arcs in Figure 5 are included in the locus of the chord midpoints. They cut off, from the region within the square, a point set of area \( \pi/4 \); this is in accordance with Holditch's theorem.

In order to make explicit the assumptions Holditch apparently made, and to clarify the conditions under which his theorem holds, we wish to carefully restate the theorem. Some definitions are needed first. A curve is a planar point set having a parametric representation of the
form $x = x(t)$, $y = y(t)$, $0 \leq t \leq 1$, where the functions $x(t)$ and $y(t)$ are continuous. A **simple closed curve** is a curve such that distinct $t$-values give distinct points of the curve with one exception: $t = 0$ and $t = 1$ give the same point. A **convex closed curve** $c$ is a simple closed curve such that, if $A$ and $B$ are two points of $c$, either no interior point of the line segment $AB$ lies on $c$ or every point of the line segment $AB$ lies on $c$. The expression “the region within a simple closed curve” has a precise meaning, coinciding with the intuitive notion. (This follows from the Jordan curve theorem. There are several proofs of this theorem in the literature. All of them are hard.)

**Theorem 1** (A Modern Version of Holditch’s Theorem). Let $c$ be a convex closed curve, and suppose that $A = A(t)$ and $B = B(t)$, $0 \leq t \leq 1$, are points that traverse $c$ counterclockwise one revolution as $t$ increases from 0 to 1, so that $AB$ is a chord of $c$ with a constant length $a + b$. Let $D = D(t)$ be the dividing point of $AB$ so that $|AD| = a$. Suppose that the direction angle $\theta = \theta(t)$, $0 \leq t \leq 1$, of the chord $AB$ (measured as in Figure 6) is an increasing continuous function of $t$ with $\theta(1) = \theta(0) + 2\pi$. Let $d$ be the locus of $D$, and assume that $d$ is a simple closed curve. Then the area of the region between $c$ and $d$ is equal to $\pi ab$.

We do not give a separate proof of this theorem here, since it is a special case of our more general result proved in the next section. It should be pointed out that the existence of the function $\theta(t)$ in the theorem seems essential [2].

**Holditch’s Theorem. A Generalized Version.**

Although the assumptions that $c$ is convex and $d$ is a simple closed curve seem essential to Holditch’s result, we can relax these requirements and still obtain a formula which measures the “area” between the two curves. Holditch’s theorem is then just a special case. Recall that a function $f(t)$ is of **bounded variation** on an interval $[u, v]$ if there is a positive number $M$ such that for every partition $\{u = t_0 < t_1 < \cdots < t_n = v\}$, $M$ is an upper bound for the sum

$$\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|.$$

**Theorem 2** (A Generalization of Holditch’s Theorem). Let $\alpha$ be a closed rectifiable curve with parametric representation $x = x(t)$, $y = y(t)$, $0 \leq t \leq 1$. Let $\theta = \theta(t)$, $0 \leq t \leq 1$, be a continuous function of bounded variation with $\theta(1) = \theta(0) + n \cdot 2\pi$, $n$ denoting an integer. Let $a$, $b$ be positive numbers. Let point $A = A(t)$, $0 \leq t \leq 1$, traverse $\alpha$, and for each $t$, let $B = B(t)$ be the point such that $AB$ is a line segment with length $a + b$ and direction angle $\theta = \theta(t)$. Let $D = D(t)$ be the point of $AB$ at the distance $a$ from $A$. Denote by $\beta$ and $\delta$ the curves traced out by $B$ and $D$ respectively, as $A$ traverses $\alpha$. Set

$$I_\alpha = \int_{\alpha} x \, dy, I_\beta = \int_{\beta} x \, dy, I_\delta = \int_{\delta} x \, dy.$$  

Then

$$I_\delta = \frac{b}{a+b} I_\alpha + \frac{a}{a+b} I_\beta - n\pi \cdot ab.$$  

Refer to Figure 7 for an illustration of the conditions given in the theorem. The assumptions (that $\alpha$ is rectifiable and that $\theta$ is of bounded variation) imply that $\beta$ and $\delta$ are also rectifiable, hence $I_\alpha$, $I_\beta$, $I_\delta$ are defined. The condition $\theta(1) = \theta(0) + n \cdot 2\pi$ obviously has the effect that the line segment $AB$ comes back to the starting position when the point $A$ has gone around the curve $\alpha$. The integer $n$ is known as the **winding number** of $AB$ [1]. If $\alpha$ is a simple curve, traversed counterclockwise, then Green’s theorem says that $I_\alpha$ is the area of the region within $\alpha$; similarly for $I_\beta$ and $I_\delta$. If $\alpha$ is traversed clockwise, then $-I_\alpha$ is the corresponding area.

If $\alpha$ and $\beta$ are the same curve (as in Holditch’s original formulation) then Theorem 2 says

$$I_\delta = I_\alpha - n\pi \cdot ab.$$
If also \( n = 1 \), then

\[
I_{\theta} = I_{\alpha} - \pi ab. \tag{1}
\]

In particular, when \( \alpha \) is a simple closed curve, \( n = 1 \), so equation \( (1) \) implies Theorem 1. However, this last equation gives more; for example, retrograde motions are allowed when \( A \) and \( B \) go around \( \alpha \). (Retrograde motion means that as the chord \( AB \) traverses \( \alpha \), one endpoint is forced to reverse its direction of travel and retrace portions of its path on \( \alpha \) in order to allow the other endpoint to continue to travel around \( \alpha \).) Retrograde motions occur in Figure 3; also in Figure 4 if \( a = b \geq 1 \).

In the following proof, our operation on line integrals may be unusual. However, the operation is legitimate, for all functions under consideration can be uniformly approximated by continuously differentiable functions. Hence, we use properties of line integrals that follow from analogous properties of the ordinary integral of a continuous function on a closed interval.

**Proof of Theorem 2.** We have

\[
I_{\beta} = \int_{\theta} x \, dy = \int_{\theta} (x + b \cdot \cos \theta) \, d(y + b \cdot \sin \theta).
\]

Set

\[
I_1 = \int_{\theta} x \, d \left( \sin \theta \right) + \cos \theta \, dy \quad \text{and} \quad I_\theta = \int_{\theta} \cos \theta \, d \left( \sin \theta \right).
\]

Then

\[
I_{\beta} = I_{\theta} + b \cdot I_1 + b^2 I_\theta.
\]

Analogously

\[
I_{\alpha} = I_{\theta} - a \cdot I_1 + a^2 I_\theta.
\]

It follows that

\[
b \cdot I_{\alpha} + a \cdot I_{\beta} = (a + b)I_{\theta} + ab(a + b)I_\theta.
\]

Now observe that

\[
I_\theta = \int_{\theta} \cos^2 \theta \, d\theta = n \pi.
\]

The conclusion follows.
Some Applications to Problems.

Theorem 2 can be applied to solve several interesting problems.

**PROBLEM.** Find the area swept out by the point $D$ on the piston rod in Figure 8(a).

**SOLUTION.** The point $A$ travels up and down with the motion of the piston, so the curve $\alpha$ which it traces is rectifiable. Theorem 2 gives, in the notation in Figure 8(b),

$$I_\alpha = \frac{b}{a+b} \cdot 0 + \frac{a}{a+b} \cdot \pi r^2 - 0 \cdot ab = \frac{a}{a+b} \cdot \pi r^2,$$

where $r$ is the radius of the circle about which $B$ traces.

**EXAMPLE.** Assume that the orbit of the earth, $E$, around the sun, $S$, is an ellipse, that the orbit of the moon, $M$, around $E$ is a circle, and that $P$ is a particle, always situated at the point where the attractions from $E$ and $M$ cancel (see Figure 8(c)). Assume that $S$, $E$, $M$, $P$ always are in one plane and that the motion has the period 19 years = 254 sidereal months. Then Theorem 2 is applicable with $I_\delta$ equal to 19 times the area of the ellipse (the interpretation of $I_\delta$ and $I_\beta$ is a little more complicated) and $n = 254$. (In reality,

$$19 \times \frac{1}{1} \text{ sidereal year} = 19 \times \frac{365.25636}{27.32165} \text{ days} = 254.006.$$ 

Nor are the other assumptions quite realistic.)

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![Figure 8](image-url)
PROBLEM (Suggested by John Reay, Western Washington University). Let a triangle be given, along with a line segment shorter than any altitude of the triangle. Let point D divide the line segment into lengths a, b, and let the line segment glide with its end points on the triangle. Assume the line segment is so short that $d$, the locus of $D$, is a simple curve (consisting of three curved parts and three line segments). Find the area $A$ of the point set between the triangle and $d$ (i.e., the sum of the areas of the regions shaded in FIGURE 9(a)).

SOLUTION. Note that when the line segment goes around the triangle, each of its end points makes some retrograde motion whenever a vertex of an acute angle is traversed, hence Theorem 1 is not applicable. Theorem 2, on the other hand, covers this situation. It shows that the area $A$ is $\pi ab$.

We wish to give a separate solution for this problem, since the ideas involved have an interest of their own. Assume $a + b = 1$, which is no essential restriction. Let $A_\phi$ denote the area of the left-hand shaded region in FIGURE 9(a), and let $d_\phi$ be its boundary. In FIGURE 9(b) we show one position of the line segment $AB$ as it glides around the triangle, near the vertex angle $\phi$. Using the notation in FIGURE 9(b), we have the following equations.

$$x_1 = \cos \theta + \sin \theta \cot \phi, \quad x_2 = \sin \theta \cot \phi$$
$$y_1 = 0, \quad y_2 = \sin \theta$$
$$x_D = a \cdot x_1 + b \cdot x_2 = a \cos \theta + \sin \theta \cot \phi$$
$$y_D = a \cdot y_1 + b \cdot y_2 = b \sin \theta$$

$$A_\phi = \int_{d_\phi} xydy$$
$$= \int_0^{\phi} (a \cos \theta + \sin \theta \cot \phi) b \cos \theta d\theta + \int_0^{b \sin \phi} y \cot \phi dy.$$

A computation of this expression gives

$$A_\phi = \frac{ab}{2} (\pi - \phi).$$

For a triangle whose angles have the measures $\phi$, $\gamma$, $\psi$ we then get, defining $A_\gamma$ and $A_\psi$ analogously to $A_\phi$,

$$A = A_\phi + A_\gamma + A_\psi = \frac{ab}{2} ((\pi - \phi) + (\pi - \gamma) + (\pi - \psi)) = \pi ab.$$

Hence, the claim in Holditch's theorem also holds here.

We note that elimination of $\theta$ between the equations for $x_D$ and $y_D$ shows that the curved arcs in FIGURE 9(a) are arcs of ellipses.
We will call the angles $\pi - \phi, \pi - \gamma, \pi - \psi$ (see Figure 10) the polar angles of the triangle. The computation of the area $A$ in the above proof can be replaced by the observation that these polar angles have the sum $2\pi$ (for the shaded regions in Figure 10 can be translated to form a circular disk). It can be shown in an analogous fashion that the formula $A = \pi ab$ holds when the triangle is replaced by an arbitrary convex polygon (and $a + b$ is sufficiently small).

**Problem** (Suggested by a referee). Take $\alpha$ to be a Reuleaux triangle of constant width $w$ (the Reuleaux triangle consists of three circular arcs with centers at the vertices of an equilateral triangle whose side has length $w$). Take $\delta$ to be the curve traced out by the midpoint $D$ of a sliding chord of length $w$ (see Figure 11). Apply Theorem 2 to determine the area $A$ of the region enclosed by $\delta$.

**Solution.** Define $I_\alpha$ and $I_\delta$ as in Theorem 2, assuming $\alpha$ is traversed counterclockwise. Then $I_\alpha$ is the sum of the areas of the three segments and the equilateral triangle in Figure 11(a):

$$I_\alpha = 3 \left( \frac{1}{6} \pi w^2 - \frac{1}{4} w^2 \sqrt{3} \right) + \frac{1}{4} w^2 \sqrt{3} - \frac{1}{2} w^2 (\pi - \sqrt{3}).$$

Further, $I_\delta = -2A$, for the interior "triangle" in Figure 11(b) is traversed clockwise two times. Equation (1) following Theorem 2 gives

$$-2A = I_\delta = I_\alpha - \pi \left( \frac{w}{2} \right)^2 = \frac{1}{2} w^2 (\pi - \sqrt{3}) - \frac{1}{4} \pi w^2,$$

and hence

$$A = \frac{1}{8} w^2 (2\sqrt{3} - \pi).$$

The area $A$ can of course also be determined by subtracting the areas of three segments from the area of a triangle. On the other hand, $A$ cannot be determined by applying the original Holditch theorem.

Is there a Holditch Theorem in $\mathbb{R}^3$?

Green's theorem (which shows that the area within a simple closed rectifiable curve $\alpha$, traversed counterclockwise, is $\int_\alpha x \, dy$) is an essential tool in our discussion of Theorem 2. It is known that Stokes' theorem in many contexts is a natural counterpart in $\mathbb{R}^3$ (in $xyz$-space) of Green's theorem in $\mathbb{R}^2$ (in the $xy$-plane). We might hope that Stokes' theorem and the technique in the proof of Theorem 2, applied to surfaces $\alpha, \beta, \delta$ in $\mathbb{R}^3$, produce a formula such as
Theorem

where $V_b = \int_0^b x \, dy \, dz$ under certain assumptions is the volume within a closed surface (and similarly for $V_\alpha, V_\beta$), $n$ is some “winding number” of a vector-valued function $AB$ in $R^3$, where $|AB|$ is constant, and $4\pi/3$ is the volume of the unit sphere. Unfortunately, this idea does not lead to the conjectured result. The following example demonstrates this.

Let $I_\alpha, I_\beta, I_\delta$ be the areas within the circles $\alpha, \beta, \delta$ respectively in Figure 12. Then (cf. Theorem 2),

$\frac{b}{a+b}I_\alpha - \frac{a}{a+b}I_\beta = \pi r^2 - \frac{b}{a+b} \pi (r-a)^2 - \frac{a}{a+b} \pi (r+b)^2 = -\pi \frac{a^2b + ab^2}{a+b} = -\pi ab.$

Rotate the entire figure around the x-axis, and let $V_\alpha, V_\beta, V_\delta$ denote the volumes within the spheres generated by $\alpha, \beta, \delta$ respectively. Then

$\frac{3}{4\pi} \left( V_\delta - \frac{b}{a+b} V_\alpha - \frac{a}{a+b} V_\beta \right) = r^3 - \frac{b}{a+b} (r-a)^3 - \frac{a}{a+b} (r+b)^3$

$= \frac{b}{a+b} (3r^2a - 3ra^2 + a^3) - \frac{a}{a+b} (3rb^2 + 3rb^2 + b^3)$

$= -\frac{3a^2b - 3ab^2}{a+b} r + \frac{ab(a^2 - b^2)}{a+b} = -3abr + ab(a-b).$

This last expression varies with $r$, in contrast to the situation in $R^2$ where Holditch’s theorem yields a constant value. Hence there is probably no Holditch theorem for $R^3$. An analogous argument indicates that there is probably no Holditch theorem in $R^n$ for $n > 3$.

References

[2] Arne Broman, Holditch’s theorem is somewhat deeper than Holditch thought in 1858, Nordisk Matematisk Tidskrift, vol. 27, 1979, pp. 89–100. In this article (in Swedish), the author gives a sufficient condition for the existence of such a function $\theta(t)$: Assume that $h$ is the largest real number such that every circle with radius less than $h$ and center on $c$ intersects $c$ at precisely two points; then $a+b<h$ is sufficient. In addition, it follows from the other assumptions that $d$ is a simple closed curve. The proof is rather long.