STABILITY OF STANDING WAVES FOR SOME NONLINEAR SCHröDINGER EQUATIONS

J.B. McLeod  
Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260

C.A. Stuart  
IACS-FSB, Section de Mathématiques, École Polytechnique Fédérale de Lausanne, CH-1015, Lausanne, Switzerland

W.C. Troy  
Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260

Abstract. The paper concerns the monotonicity with respect to $\lambda$ of the $L^2$-norm of the branch of positive solutions of the nonlinear eigenvalue problem

$$u''(x) + g(x, u(x)^2) u(x) + \lambda u(x) = 0, \quad x \in \mathbb{R}, \quad \lim_{|x|\to\infty} u(x) = 0.$$  

For the particular case $g(x) = p(x) + s^\sigma$, with $\sigma > 0$ and $p(x)$ an even function, decreasing for $x > 0$ and with $p(\infty) = 0$, the main theorem implies that the $L^2$-norm decreases as we increase $\lambda$ if $\sigma \leq 2$. It is also shown that this is no longer true if $\sigma > 2$. The result has implications for the orbital stability of standing waves of the nonlinear Schrödinger equation.

1. INTRODUCTION

This paper concerns the monotonicity with respect to $\lambda$ of the $L^2$-norm of the branch of positive solutions of the nonlinear eigenvalue problem

$$u''(x) + g(x, u(x)^2) u(x) + \lambda u(x) = 0 \quad \text{for} \quad x \in \mathbb{R}, \quad \lim_{|x|\to\infty} u(x) = 0.$$  

Our hypotheses about the function $g$ are set out below, but to present our work briefly we concentrate here on the special case where

$$g(x, s) = p(x) + s^\sigma.$$  

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with \( \sigma > 0 \) and \( p \) an even, continuously differentiable function with \( p'(x) \leq 0 \) for \( x \geq 0 \) and such that \( p(0) > 0 \) and \( \lim_{x \to \infty} p(x) = 0 \). From Theorems 1 and 2 of [1] it follows that there exist \( \Lambda < 0 \) and a \( C^1 \)-function \( u : (\infty, \Lambda) \to H^2(\mathbb{R}) \) such that \( (\lambda, u(\lambda)) \) satisfies (1.1) and (1.2) for all \( \lambda < \Lambda \). Furthermore, \( u(\lambda) \) is even with \( u(\lambda) > 0 \) and \( \frac{d}{dx} u(\lambda) < 0 \) for \( x > 0 \), and all positive solutions of (1.1)–(1.2) lie on the curve \( \{ (\lambda, u(\lambda)) : \lambda \in (-\infty, \Lambda) \} \). It is also shown that the maximum of \( u(\lambda) \) is strictly decreasing in \( \lambda \) since \( \frac{d}{dx} u(\lambda)(0) < 0 \) for all \( \lambda < \Lambda \) and that \( \int_{-\infty}^{\infty} u(\lambda)^2 dx \to 0 \) as \( \lambda \to \Lambda \), but [1] contains no information about the monotonicity of the \( L^2 \)-norm of \( u(\lambda) \).

Our main result in the present paper shows that, for \( 0 < \sigma < 2 \), \( \int_{-\infty}^{\infty} u(\lambda) \frac{d}{d\lambda} u(\lambda) dx < 0 \) for all \( \lambda < \Lambda \),

\[ (1.4) \]

and hence \( \int_{-\infty}^{\infty} u(\lambda)^2 dx \) is a strictly decreasing function of \( \lambda \). We also show in an appendix that, when \( \sigma > 2 \), \( \int_{-\infty}^{\infty} u(\lambda)^2 dx \to 0 \) as \( \lambda \to -\infty \), and hence \( \int_{-\infty}^{\infty} u(\lambda)^2 dx \) is not a monotone function of \( \lambda \) in this case.

Our interest in this monotonicity of the branch of positive solutions of (1.1)–(1.2) stems from the fact that it implies the orbital stability of these solutions when viewed as standing waves for the nonlinear Schrödinger equation

\[ i\partial_t v + \partial^2_{xx} v + g(x, |v|^2)v = 0 \quad \text{for} \quad x \in \mathbb{R} \quad \text{and} \quad t \geq 0, \]

where \( v = v(x, t) \) is complex-valued. Setting \( V_\lambda(x, t) = e^{-i\lambda t} u(\lambda)(x) \) for \( \lambda < \Lambda \), we observe that \( V_\lambda \) is a periodic solution of (1.5) whose orbit is \( \{ e^{i\theta} u(\lambda) : \theta \in \mathbb{R} \} \). A standing wave of this kind is said to be orbitally stable in \( X = H^1(\mathbb{R}) \) if, for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that

\[
\sup_{t \geq 0} \inf_{\theta \in \mathbb{R}} ||v(\cdot, t) - e^{i\theta} u(\lambda)||_X < \epsilon \quad \text{whenever} \quad ||v(\cdot, 0) - u(\lambda)||_X < \delta.
\]

(This condition implicitly includes the requirement that, for any initial condition close to \( u(\lambda) \) in \( X \), the initial-value problem for (1.5) has a solution \( v(\cdot, t) \in X \) defined for all \( t \geq 0 \). For (1.5) with (1.3), this is true provided that \( 0 < \sigma < 2 \).) Necessary and sufficient conditions for orbital stability in this sense were established in [2] for a general class of infinite-dimensional Hamiltonian systems. In the context of (1.5), the main assumption in [2] is that the linearization of (1.1)–(1.2) at \( u(\lambda) \) has exactly one negative eigenvalue and that it is simple. Under our hypotheses about \( g \) this follows immediately from the results in [1], since it is shown there that the linearization of (1.1)–(1.2) is invertible for all \( \lambda < \Lambda \). Since the linearization at the bifurcation point \( (\Lambda, 0) \) has 0 as its lowest eigenvalue, it follows by an elementary
comparison and continuity that the linearization has exactly one negative eigenvalue for all $\lambda < \Lambda$. (Of course all eigenvalues are simple since (1.1) is of second order.) In the context of (1.5), part (A) of Theorem 3 in [2] can be stated as follows. If

$$\int_{-\infty}^{\infty} u(\lambda) \frac{d}{d\lambda} u(\lambda) dx \neq 0,$$

then the solution $V_\lambda$ is orbitally stable if and only if (1.4) holds, whence our interest in finding conditions on $g$ that ensure that (1.4) is satisfied for every $\lambda < \Lambda$.

Our assumptions about $p$ and $g$ are set out at the beginning of the next section, where $g(x, s^2) s$ is written as $F(x, s)$. The conditions (ii) and (iii) mean that $g(x, s^2)$ is nonincreasing in $x$ and nondecreasing in $s$. In view of the evenness of the positive solutions of (1.1)–(1.2) mentioned above, it is sufficient for us to deal with (1.1) for $x > 0$ and to replace (1.2) by $u'(0) = \lim_{x \to -\infty} u(x) = 0$. Hence, our assumptions about $p$ and $F$ refer only to $x \geq 0$, but it is a trivial matter to add conditions (essentially, $p$ is continuously differentiable with $p'(0) = 0$ and $\partial_x F(0, s) = 0$) ensuring the existence of an even $C^1$-extension to $\mathbb{R}$ so as to recover the setting for the existence results from [1] which we have cited above. Under these more general conditions, $\lim_{s \to -\infty} F(x, s)/s$ may be finite or infinite and, referring to the conditions (L1) and (L2) of [1], we see that now the branch of positive solutions of (1.1)–(1.2) is defined for $\lambda$ in an interval $(\lambda^*, \Lambda)$ where $\lambda^* > -\infty$ when $\lim_{s \to -\infty} F(x, s)/s$ is finite and $\lambda^* = -\infty$ when this limit is infinite.

The method of proof is by continuation. If we set

$$\phi = du/d\lambda,$$

then we can show that, if $\lambda$ is sufficiently close to the bifurcation value $\Lambda$, then $\phi/u$ is strictly increasing as a function of $x$ (Lemma 2.1 below). However, as we vary $\lambda$, this property of $\phi/u$ cannot be lost (Lemma 2.2), and so it is valid everywhere on the branch of solutions. This fact, introduced into some manipulations on the equation (1.1), leads to the final result.

It should be remarked that the continuity argument can also be carried out starting at the other end of the branch. For example, if we take (1.1) and (1.3) with $\sigma = 1$, so that the equation is

$$u'' + p(x) u + u^3 + \lambda u = 0$$

and $\lambda \to -\infty$ on the branch, then the appropriate scaling for large $|\lambda|$ is

$$t = (-\lambda)^{1/2} x, \quad u = (-\lambda)^{1/2} w,$$
so that
\[ w'' - w + w^3 = \frac{1}{\lambda} p(t/\sqrt{-\lambda})w, \quad w'(0) = 0, \]
and as \(|\lambda| \to \infty\), \( w \to w_\infty \), where
\[ w''_\infty - w_\infty + w^3_\infty = 0, \quad w_\infty = \sqrt{2}/\cosh t. \]
We can similarly write down the equation for \( \psi = dw/d\lambda \), and its limit as \(|\lambda| \to \infty\), and from these two equations deduce that \( \psi/w \) is strictly increasing as a function of \( x \), as required. (The same scaling is used in the appendix.)

Finally, we show in an appendix that, if the conditions of Theorem 2.1 below are not met, then monotonicity of \( \int_{-\infty}^{\infty} t^2 dx \) may not hold, specifically for (1.1) and (1.3) with \( \sigma > 2 \).

2. Assumptions and statement of results

We consider the boundary-value problem
\[
\begin{align*}
-u'' - p(x)u - F(x, u) &= \lambda u, \quad 0 < x < \infty, \\
u'(0) &= 0, \\
u(\infty) &= 0,
\end{align*}
\]
where
(i) \( p(x) \) is nonincreasing for \( 0 \leq x < \infty \), with \( p(0) > 0, p(\infty) = 0 \);
(ii) \( F(x, u) \) is continuously differentiable in both \( x \) and \( u \) for \( 0 \leq x < \infty \) and \( 0 < u < \infty \), \( F(x, u) = o(u) \) as \( u \to 0 \), uniformly in \( x \);
(iii) \( F_u - (F/u) \) is positive nonincreasing as \( x \) increases and \( u \) decreases, with (necessarily from (ii)) \( F_u - (F/u) \to 0 \);
(iv) \( (5F - uF_u + 2xF_x)/(uF_u - F) \) is also nonnegative and nonincreasing in \( u \). (It is clear that the assumptions on \( F \) are satisfied if, for example,
\[
F(x, u) = u^k, \quad 1 < k \leq 5.
\]

**Theorem 2.1.** Suppose that \( p, F \) satisfy assumptions (i)-(iv). If, for the boundary value problem (2.1)-(2.3), there exists, for \(-\infty \leq \lambda^* < \lambda < \Lambda < 0 \), a branch of positive solutions \( u(x, \lambda) \) with \( u'(x, \lambda) \) \( u(x, \lambda) \to 0 \) as \( \lambda \uparrow \Lambda \), then
\[
\int_{-\infty}^{\infty} u^2(x, \lambda) \, dx
\]
deincreases as \( \lambda \) increases from \( \lambda^* \) to \( \Lambda \).
(The introduction discusses the conditions under which such a branch of positive solutions does indeed exist.)

The proof depends on considering

$$\phi = \partial u(x, \lambda) / \partial \lambda,$$

which, from (2.1), satisfies

$$-\phi'' - p(x) \phi - F_u(x, u) \phi = u + \lambda \phi,$$

$$\phi'(0) = 0,$$

$$\phi(\infty) = 0.$$  \hfill (2.4)

The asymptotics as $x \to \infty$ are easily calculated from (2.1) and (2.4). Thus both $u$ and $\phi$ vanish exponentially fast as $x \to \infty$. Multiplying (2.1) by $\phi$ and (2.4) by $u$ and subtracting, we have

$$(\phi'u - \phi'u') + (uF_u - F)\phi = -u^2,$$

so that, integrating and imposing the boundary conditions at 0, we obtain

$$\phi'u - \phi'u = -\int_0^x \{u^2 + (uF_u - F)\phi\} dt,$$

and, letting $x \to \infty$,

$$\int_0^{\infty} \{u^2 + (uF_u - F)\phi\} dt = 0.$$  \hfill (2.9)

In view of condition (iii) on $F$, it is clear that we cannot have everywhere $\phi \geq 0$. The necessary facts about the behaviour of $\phi$ are contained in the following two lemmas.

**Lemma 2.1.** If $\lambda$ is sufficiently close to $\Lambda$, $\lambda < \Lambda$, then $\phi(x, \lambda)$ has precisely one zero, at $x_0(\lambda)$. Further,

(a) $\phi(0, \lambda) < 0$, $\phi'(x, \lambda) > 0$ for $0 < x \leq x_0$,

(b) $\phi/u$ is strictly increasing for $x > 0$.

**Lemma 2.2.** Properties (a) and (b) of Lemma 2.1 hold for all $\lambda < \Lambda$.

In the next section, we use Lemmas 2.1 and 2.2 to prove the theorem. Lemmas 2.1 and 2.2 are proved thereafter.
3. Proof of the theorem

We prove the theorem by continuation. It is clear that, for at least a sequence of values \( \{ \lambda_n \} \), \( \lambda_n \uparrow \Lambda \), we have

\[
\frac{d}{d\lambda} \int_0^\infty u^2(x, \lambda) dx < 0; \text{ i.e., } \int_0^\infty u \phi dx < 0,
\]

since the first integral is zero in the limit as \( \lambda \to \Lambda \).

Now decrease \( \lambda \) from such a value \( \lambda_n \), and suppose for contradiction that there is a first \( \lambda_0 \), say \( \lambda_0 \), for which

\[
\int_1^0 u \phi dx = 0.
\]

(3.1)

If we establish the contradiction, then we have proved the theorem.

Since (3.1) cannot be true if \( \phi \leq 0 \), we must suppose that \( \phi/u \) has a zero, say at \( x_0 \). At \( \lambda = \lambda_0 \), consider the function

\[
g(x) = xu\phi - 2 \int_0^x u \phi dt.
\]

Since, by Lemma 2.1, \( u\phi \) is first negative and then positive, (3.1) implies that, for all \( x > 0 \),

\[
\int_0^x u \phi dt < 0,
\]

and so \( g(x) > 0 \) for \( \phi \geq 0 \), i.e., for \( x \geq x_0 \). Also,

\[
g' = -u\phi + x(u\phi)'.
\]

In \( (0, x_0) \) we have \( \phi < 0, (\phi/u)' > 0 \), so that \( (u\phi)' > 0, g' > 0 \), and, since \( g(0) = 0, g > 0 \). Hence, \( g(x) > 0 \) for \( 0 < x < \infty \).

Multiplying (2.1) by \( u \) and integrating, we have

\[
\int_0^\infty u'^2 dx - \int_0^\infty pu'^2 dx - \int_0^\infty uF dx = \lambda \int_0^\infty u^2 dx,
\]

(3.2)

and multiplying by \( xu' \) and integrating by parts, we have

\[
\int_0^\infty u'^2 dx + \int_0^\infty (xp' + p)u'^2 dx - 2 \int_0^\infty xu'F dx = -\lambda \int_0^\infty u^2 dx.
\]

(3.3)

(If \( p \) is not differentiable, the second integral above is to be understood as a Stieltjes integral.) But

\[
\frac{d}{dx} \left( x \int_0^{u(x)} F(x, s) ds \right) = \int_0^{u(x)} F(x, s) ds + xu'F(x, u) + x \int_0^{u(x)} F_x(x, s) ds,
\]
so that (3.3) becomes
\[
\int_0^\infty u^2 \, dx + \int_0^\infty (xp' + p)u^2 \, dx + 2 \int_0^\infty \left( \int_0^u F(x, s) \, ds \right) \, dx \\
+ 2 \int_0^\infty \left( x \int_0^u F_x(x, s) \, ds \right) \, dx = -\lambda \int_0^\infty u^2 \, dx.
\] (3.4)

Subtracting (3.2) and (3.4) gives
\[
\int_0^\infty (xp' + 2p)u^2 \, dx + \int_0^\infty uF \, dx + 2\lambda \int_0^\infty u^2 \, dx + 2 \int_0^\infty \left( \int_0^u F(x, s) \, ds \right) \, dx \\
+ 2 \int_0^\infty \left( x \int_0^u F_x(x, s) \, ds \right) \, dx = 0.
\]
Now differentiate with respect to \( \lambda \), so that
\[
2 \int_0^\infty (xp' + 2p)u \phi \, dx + \int_0^\infty uF \phi \, dx + \int_0^\infty uF_u \phi \, dx + 2 \int_0^\infty u^2 \, dx + 4\lambda \int_0^\infty u \phi \, dx \\
+ 2 \int_0^\infty F \phi \, dx + 2 \int_0^\infty xF_x \phi \, dx = 0;
\]
i.e., from (2.9),
\[
2 \int_0^\infty (xp' + 2p)u \phi \, dx + 4\lambda \int_0^\infty u \phi \, dx + 5 \int_0^\infty F \phi \, dx - \int_0^\infty uF_u \phi \, dx \\
+ 2 \int_0^\infty xF_x \phi \, dx = 0,
\]
\[
\int_0^\infty (xp' + 2p)u \phi \, dx + 2\lambda \int_0^\infty u \phi \, dx + \frac{1}{2} \int_0^\infty (5F - uF_u + 2xF_x) \phi \, dx = 0.
\] (3.5)
Now, again from (2.9),
\[
\int_0^\infty (uF_u - F) \phi \, dx < 0,
\]
and since, by hypothesis, \( G(x, u) \equiv \frac{5F - uF_u + 2xF_x}{uF_u - F} \) is positive decreasing, we have
\[
\int_0^\infty (5F - uF_u + 2xF_x) \phi \, dx = \int_0^\infty 5F - uF_u + 2xF_x \left\{ (uF_u - F) \phi \right\} \, dx \\
= \int_0^\infty \left\{ \frac{5F - uF_u + 2xF_x}{uF_u - F} (x) \right\} \left\{ (uF_u - F) \phi \right\} \, dx \\
+ \frac{5F - uF_u + 2xF_x}{uF_u - F} (x_0) \int_0^\infty (uF_u - F) \phi \, dx \leq 0.
Hence, at $\lambda = \lambda_0$, where we are assuming for contradiction that
\[
\int_0^\infty u\phi \, dx = 0,
\]
we have the necessary contradiction, from (3.5), if
\[
\int_0^\infty (xp' + 2p)u\phi \, dx < 0.
\]
But
\[
\int_0^\infty (xp' + 2p)u\phi \, dx
= \int_0^\infty xp'\phi \, dx + \left[2p\int_0^x u\phi \, dt\right]_0^\infty - 2\int_0^\infty p'\left(\int_0^x u\phi \, dt\right) \, dx = \int_0^\infty p'g \, dx < 0,
\]
as required.

4. Proof of Lemma 2.1

We know that $\phi$ satisfies the boundary-value problem
\begin{align*}
-\phi'' - p\phi - F_u\phi - \lambda\phi &= u, \quad \text{(4.1)} \\
\phi'(0) &= 0, \quad \phi(\infty) = 0. \quad \text{(4.2)}
\end{align*}

The solution of this problem is (given $u$) unique. For if it were not, then the homogeneous problem would have to have a nontrivial solution, $\phi_0$ say, and by the Fredholm alternative, $u$ would have to be orthogonal to $\phi_0$. But since $F_u$ is small and $\lambda$ is close to $\lambda$, the lowest eigenvalue of
\[
-\phi'' - p\phi - \mu\phi = 0
\]
(with the obvious boundary conditions), $\lambda$ must be the lowest eigenvalue of the homogenous problem and so $\phi_0$ is of one sign, contradicting $u$ orthogonal to $\phi_0$.

We now exhibit a solution of (4.1)--(4.2) possessing the properties required in Lemma 2.1. Since the solution is unique, we will then be done.

Define the solution $\phi_k$ of
\[
-\phi'' - p\phi - F_u\phi - \lambda\phi = u
\]
by $\phi'(0) = 0$, $\phi(0) = k$. We write, as in (2.7),
\[
(\phi'u - \phi'')' = -u^2\left(1 + \frac{F_u}{u}\right)\frac{\phi}{u},
\]
and restrict attention to those $k$ for which
\[ \left( F_u - \frac{F}{u} \right) \left( \frac{\phi}{u} \right)(0) < -1, \]
so that $(\phi/u)' > 0$ for $x$ small. Consider two subsets of such $k$:
\[ S_1 = \{ k : (\phi/u)' \text{ becomes negative at some finite } x \}, \]
\[ S_2 = \{ k : (\phi/u)' \text{ is strictly positive, with } \phi' \text{ ultimately positive} \}. \]
The sets are clearly disjoint, and $S_1$ clearly open. Note also that
\[ \left( \frac{\phi}{u} \right)' = \frac{\phi'u - \phi u'}{u^2} > 0 \text{ implies } \phi' > 0 \text{ when } \phi \leq 0. \]
We want to show next that $S_2$ is open. If $k_0 \in S_2$, then, for $k = k_0, (\phi/u)' > 0$
implies that $\phi$ has at most one zero. If it has no zero, then $(\phi/u)$ is negative
increasing, and so tends (as $x \to \infty$) to a finite nonpositive limit, and so
$\phi \sim -au$, say, $a \geq 0$. But the equation for $\phi$ gives, as $x \to \infty$,
\[ -\phi'' \sim u + \lambda \phi \sim \left( \lambda - \frac{1}{a} \right) \phi = \nu \phi, \text{ say,} \]
where $|\nu| > \lambda$. This implies $\phi = o(u)$, so that
\[ -\phi'' \sim u \sim -\lambda^{-1}u'', \quad \phi' \sim \lambda^{-1}u', \quad \phi \sim \lambda^{-1}u, \]
contradicting $\phi = o(u)$.
So, for $k = k_0$, there exists $x_0$ such that $\phi(x) > 0$ and $\phi'(x) > 0$ if $x > x_0$.
Thus, for $x$ sufficiently large, $\phi'' \sim (-\lambda)\phi$, so that $\phi'$ is increasing and $\phi''$
tends to a nonzero limit (possibly $\infty$). Clearly, the same holds for $k$ near
$k_0$, so that, for $k$, $\phi'$ is ultimately positive. Finally, for $k, (\frac{\phi}{u})' > 0$
in any bounded range of $x$, since that is true when $k = k_0$, and the facts $\phi' > 0$ and
$\phi > 0$ imply that it is true ultimately and so always.
This shows $S_2$ is open.
If we can prove that $S_1$ and $S_2$ are nonempty, then there exists some $k$
in neither, say $k = k^*$. For such $k^*$, since $k^* \notin S_1$, we have $(\phi/u)' \geq 0$. But in
fact, from (2.8),
\[ \left( \frac{\phi}{u} \right)' = -\frac{1}{u^2} \int_0^x u^2 \{ 1 + \left( F_u - \frac{F}{u} \right) \frac{\phi}{u} \} dt. \quad (4.3) \]
For $\phi < 0$, $\phi/u$ nondecreasing and $F_u - (F/u)$ nonincreasing imply that the
integrand is first negative (from the range of $k$ we are considering) and then
positive, and it remains positive when $\phi > 0$. So the fact that $(\phi/u)'(\infty) \geq 0$
implies $(\phi/u)' > 0$. Since $k^* \notin S_2$, it must follow that $\phi'$ is not positive
ultimately, so that there must be arbitrarily large values of $x$ where $\phi' \leq 0$. 
But \( \phi/u \) is increasing and (as before) \( \phi/u \) must have a zero, so that \( \phi/u \) is ultimately positive increasing, \( \phi \sim au \) (possibly \( a = \infty \)). In fact, \( a = \infty \), because for \( k = k^* \) we must have

\[
I(x) = \int_0^x u^2 \left\{1 + \left(F_u - \frac{F}{u}\right) \frac{\phi}{u}\right\} dt
\]

ultimately negative increasing, so that either \( I(\infty) < 0 \) or \( I(\infty) = 0 \). If \( I(\infty) < 0 \), \( (\phi/u)' \) is exponentially large, from (4.3), and certainly \( \phi/u \to \infty \). If \( I(\infty) = 0 \) and we suppose for contradiction that \( \phi/u \) tends to a finite limit, then, again from (4.3),

\[
\left(\frac{\phi}{u}\right)' \sim \frac{1}{u^2} \int_x^{\infty} u^2 dt.
\]

But \( u'' \sim -\lambda u \), \( u' \sim -\sqrt{-\lambda} u \), and \( u' \sim -\sqrt{-\lambda} u \), so that

\[
\left(\frac{\phi}{u}\right)' \sim -\frac{1}{u^2} \int_x^{\infty} \frac{u u'}{\sqrt{-\lambda}} dt = \frac{1}{2\sqrt{-\lambda}},
\]

and again \( \phi/u \to \infty \).

Thus, ultimately \( \phi'' \sim -\lambda \phi \), \( \phi > 0 \), and \( \phi'' > 0 \), and we must have \( \phi' < 0 \), since otherwise \( \phi' \) becomes positive increasing and we are in \( S_2 \). Hence, for \( k = k^* \), \( \phi(\infty) = 0 \), and we are done.

To show \( S_1 \) nonempty, we write

\[
u^2 \frac{d}{dx} = \frac{d}{dX},
\]

so that the equation for \( \phi/u \) becomes

\[
\frac{d^2}{dX^2} \left(\frac{\phi}{u}\right) + u^4 \left\{1 + \left(F_u - \frac{F}{u}\right)(0) \frac{\phi}{u}\right\} = -u^4 \phi \left\{ \left(F_u - \frac{F}{u}\right)(X) - \left(F_u - \frac{F}{u}\right)(0)\right\},
\]

or, with

\[
\psi = \frac{\phi}{u} + \frac{1}{\left(F_u - \frac{F}{u}\right)(0)},
\]

\[
T\psi \equiv \frac{d^2}{dX^2} \psi + \frac{u^4}{\left(F_u - \frac{F}{u}\right)(0)} \psi = -u^4 \phi \left\{ \left(F_u - \frac{F}{u}\right)(X) - \left(F_u - \frac{F}{u}\right)(0)\right\} < 0
\]

when \( \phi < 0 \). (This assumes that \( F_u - \frac{F}{u} \) is strictly decreasing, but the argument can be refined to deal with the case where it is only nonincreasing.)
Let \( \psi_1 \) be the solution of \( T \psi = 0 \) with \( \psi_1(0) = 1 \) and \( \psi_1'(0) = 0 \), and \( \psi_2 \) the solution with \( \psi_2(0) = 0 \) and \( \psi_2' = 1 \). Then, since \( \psi'(0) = 0 \), if we make the additional restriction that \( \psi(0) = 0 \), we have

\[
\psi = \int_0^X u^4 \phi u \left\{ \left( F_u - \frac{F}{u} \right) - \left( F_u - \frac{F}{u} \right) (0) \right\} \{ \psi_1(X) \psi_2(t) - \psi_2(X) \psi_1(t) \} dt,
\]

\[
\psi' = \int_0^X u^4 \phi u \left\{ \left( F_u - \frac{F}{u} \right) - \left( F_u - \frac{F}{u} \right) (0) \right\} \{ \psi_1'(X) \psi_2(t) - \psi_2'(X) \psi_1(t) \} dt.
\]

For small \( X \),

\[
\psi' \sim - \int_0^X u^4 \left\{ \left( F_u - \frac{F}{u} \right) - \left( F_u - \frac{F}{u} \right) (0) \right\} dt < 0.
\]

Thus, for the initial condition, say \( k = k_1 \), given by

\[
\left( F_u - \frac{F}{u} \right) \left( \frac{\phi}{u} \right) (0) = -1,
\]

we have that \( (\phi/u)' \) becomes negative initially, and that \( (\phi/u)' \) becomes negative must continue to hold for \( k \) sufficiently close to \( k_1 \), showing \( S_1 \) nonempty.

To show \( S_2 \) nonempty, we take \( (\phi/u)(0) \) very large and negative. Then, since \( -\phi > u \), we see from the equations for \( \phi \) and \( u \) that we continue to have \( -\phi > u \) for a large \( x \)-interval, during which, from (2.8), \( \phi'u - \phi u' \) becomes large and positive. So long as we continue to have \( (F_u - \frac{F}{u}) \frac{\phi}{u} < -1 \), \( \phi'u - \phi u' \) continues to grow larger, and if \( \phi \) does not cross zero, then, once \( (F_u - \frac{F}{u}) \frac{\phi}{u} > -1 \), at \( x = X \), say, the contribution to \( \phi'u - \phi u' \) does not exceed \( \int_X^\infty u^2 dt \), which is small. Thus \( \phi'u - \phi u' \) remains large for all \( x \), and

\[
(\phi/u)' > K/u^2, \quad K > 0,
\]

which certainly implies that \( \phi \) crosses zero, and when it does so, \( \phi' > K/u \). Thereafter, \( \phi'' \sim -\lambda \phi \), so that \( \phi \) grows exponentially and we are in \( S_2 \).

This completes the proof of Lemma 2.1.

5. Proof of Lemma 2.2

Suppose for contradiction that there is a first \( \lambda \), say \( \lambda_0 \), for which properties (a) and (b) fail to hold. Then, for \( \lambda = \lambda_0 \), \( \phi(0, \lambda_0) \leq 0 \) and \( \phi/u \) is increasing, although perhaps not strictly.
But we cannot have \( \phi(0, \lambda_0) = 0 \), since then \( \phi/u \) nondecreasing implies \( \phi \geq 0 \), contradicting (2.9). So \( \phi(0, \lambda_0) < 0 \), and (2.8), with \( \phi/u \) nondecreasing and \( F_u - (F/u) \) nonincreasing, implies that we can write

\[
u^2 + (uF_u - F)\phi = u^2 \left( 1 + \left[ F_u - \left( F/u \right) \right] \frac{\phi}{u} \right)
\]

so that the integrand on the right of (2.8) is nonpositive for \( x \leq X \), say, and nonnegative for \( x \geq X \), implying that \( (\phi/u)' \) is strictly positive, and so \( \phi/u \) is strictly increasing. This proves Lemma 2.2, and completes the proof of the theorem.

6. Appendix

We consider (1.1)--(1.2) where \( g \) is given by (1.3) and show that

when \( \sigma > 2 \), \( \int_{-\infty}^{\infty} u(\lambda)^2 dx \to 0 \) as \( \lambda \to -\infty \),

(6.1)

where \( \{(\lambda, u(\lambda)) : \lambda \in (-\infty, \Lambda)\} \) is the curve of positive solutions of (1.1)--(1.2) discussed in the introduction. Since we also have that

\[
\int_{-\infty}^{\infty} u(\lambda)^2 dx \to 0 \text{ as } \lambda \to \Lambda,
\]

this shows that \( \int_{-\infty}^{\infty} u(\lambda)^2 dx \) is not a monotone function of \( \lambda \) when \( \sigma > 2 \).

To deal with (1.1) as \( \lambda \) tends to \(-\infty \) we make the following change of variables:

\[
w(x) = (-\lambda)^{-\frac{1}{2\sigma}} u((-\lambda)^{-\frac{1}{2\sigma}} x).
\]

Then \( (\lambda, u) \) satisfies (1.1)--(1.2) if and only if \( (\lambda, w) \) satisfies

\[
w''(x) + w(x)^{2\sigma+1} - w(x) - q(\lambda, x)w(x) = 0, \quad (6.2)
\]

\[
\lim_{|x| \to \infty} w(x) = 0, \quad (6.3)
\]

where \( q(\lambda, x) = \frac{1}{\chi} p((-\lambda)^{-\frac{1}{2\sigma}} x) \).

Lemma 6.1. There exists \( L \) such that for every \( \lambda < L \), (6.2)--(6.3) has a positive solution \( w(\lambda) \) and

\[
\int_{-\infty}^{\infty} w(\lambda)^2 dx \to \int_{-\infty}^{\infty} w_\infty^2 dx \text{ as } \lambda \to -\infty,
\]

where \( w_\infty(x) = (1 + \sigma)^{\frac{1}{2\sigma}} \cosh^{\frac{1}{\sigma}} (\sigma x) \).
By the uniqueness of the positive solutions of (1.1)-(1.2) that is established in [1], it follows that 
\[ u(\lambda)(x) = (-\lambda)^{\frac{\sigma}{2}} w(\lambda)((-\lambda)^{\frac{1}{2}} x) \]
and hence we see that
\[ \int_{-\infty}^{\infty} u(\lambda)^2 dx = (-\lambda)^{\frac{1}{2}} \int_{-\infty}^{\infty} w(\lambda)((-\lambda)^{\frac{1}{2}} x)^2 dx = (-\lambda)^{\frac{1}{2} - \frac{1}{2}} \int_{-\infty}^{\infty} w(\lambda)(t)^2 dt. \]
Thus,
\[ (-\lambda)^{\frac{1}{2} - \frac{1}{2}} \int_{-\infty}^{\infty} u(\lambda)^2 dx \to \int_{-\infty}^{\infty} w_\infty^2 dx \text{ as } \lambda \to \Lambda, \]
proving (6.1).

**Proof of Lemma 6.1.** Our claim follows from minor modifications of the arguments used to prove Theorem 1 and Lemma 3.1 in [3]. The existence of a solution \( w(\lambda) \) could be reduced to an application of the contraction-mapping theorem, but it is quicker to use the implicit-function theorem. As in [1], let
\[ X = \{ w \in H^2(0, \infty) : w'(0) = 0 \} \]
and consider the function \( G : \mathbb{R} \times X \to Y \) defined by
\[
G(t, w) = \begin{cases} 
  w'' + |w|^{2\sigma} \frac{w - w - q(-\frac{1}{|t|}, \cdot)w}{2} & \text{if } t \neq 0 \\
  w'' + |w|^{2\sigma} \frac{w - w}{2} & \text{if } t = 0.
\end{cases}
\]
Since \( |q(-\frac{1}{|t|}, x)| \leq |t|p(0) \) for all \( x \), it follows easily from the proof of Lemma 6 in [1] that \( G : \mathbb{R} \times X \to Y \) is continuous and that it admits a partial derivative \( D_w G(t, w) \) in the sense of Fréchet, where
\[
D_w G(t, w) v = \begin{cases} 
  v'' + (2\sigma + 1) |v|^{2\sigma} \frac{v - v - q(-\frac{1}{|t|}, \cdot)v}{2} & \text{if } t \neq 0 \\
  v'' + (2\sigma + 1) |v|^{2\sigma} \frac{v - v}{2} & \text{if } t = 0
\end{cases}
\]
for \( w, v \in X \). Furthermore, the bounded linear operator \( D_w G(t, w) : X \to Y \) depends continuously on \((t, w)\).

Clearly \( w_\infty \in X \) and \( G(0, w_\infty) = 0 \). The proof of Theorem 1 in [3] shows that \( D_w G(0, w_\infty) : X \to Y \) is an isomorphism (in [3], \( X = W^{2,1}(0, \infty) \) and \( Y = L^1(0, \infty) \), but everything remains true in our setting), and hence, by the implicit-function theorem, there exist \( T > 0 \) and \( W \in C((-T, T), X) \) such that \( W(0) = w_\infty \) and \( G(t, W(t)) = 0 \) for all \( t \in (-T, T) \). Setting \( w(\lambda) = W(\frac{1}{\lambda}) \) for \( \lambda < L = -\frac{1}{T} \), we see that \( w(\lambda) \in X, w(\lambda) \) satisfies
\[
w'' + |w|^{2\sigma} \frac{w - w - q(\lambda, \cdot)w}{2} = 0,
\]
and $w(\lambda)$ converges to $w_\infty$ in $X$ as $\lambda \to -\infty$. Taking the even extension of $w(\lambda)$ to $\mathbb{R}$, only the positivity of $w(\lambda)$ remains to be proved. Since $p$ is nonincreasing,

$$1 + q(\lambda, x) \geq 1 + \frac{1}{\lambda} p(0) \geq \frac{1}{2} \quad \text{for} \quad x \geq 0 \quad \text{and} \quad \lambda \leq -2p(0).$$

Hence, there exists $\xi > 0$ such that

$$1 + q(\lambda, x) - |w_\infty(x)|^{2\sigma} \geq \frac{1}{4} \quad \text{for} \quad x \geq \xi \quad \text{and} \quad \lambda \leq -2p(0).$$

But $w(\lambda)$ converges uniformly to $w_\infty$ on $[0, \infty)$ as $\lambda \to -\infty$, and so we may suppose that $|w(\lambda)(x)|^{2\sigma} \leq |w_\infty(x)|^{2\sigma} + \frac{1}{8}$ for all $\lambda < L$. Thus, we have that

$$1 + q(\lambda, x) - |w(\lambda)(x)|^{2\sigma} \geq \frac{1}{8} \quad \text{for} \quad x \geq \xi \quad \text{and} \quad \lambda < L,$$

and consequently, for $w = w(\lambda)$,

$$w(x)w'(x) + \int_0^\infty w'(y)^2 dy = -\int_0^\infty w(y)^2 \{1 + q(\lambda, y) - |w(y)|^{2\sigma}\} dy < 0$$

for $x \geq \xi$ and $\lambda < L$. In particular, $w(\lambda)(x) \neq 0$ for $x \geq \xi$ and $\lambda < L$. But $w_\infty(x) \geq w_\infty(\xi)$ for $0 \leq x \leq \xi$ and $w(\lambda)$ converges uniformly to $w_\infty$ on $[0, \xi]$ as $\lambda \to -\infty$ so that we may assume that $w(\lambda)(x) \geq \frac{1}{2} w_\infty(\xi)$ for $x \in [0, \xi]$ and $\lambda < L$. Since $w(\lambda)$ is continuous on $[0, \infty)$, this establishes the positivity of $w(\lambda)$ on $[0, \infty)$ for all $\lambda < L$.

**References**

