New Singular Standing Wave Solutions Of The Nonlinear Schrodinger Equation

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Abstract

We prove existence, and asymptotic behavior as $r \to \infty$, of a family of singular solutions of

$$(1) \qquad y'' + \frac{2}{r} y' + y|y|^{p-1} - y = 0, \ \ 0 < r < \infty, \ \ 2 < p \leq 3,$$

which satisfy $\lim_{r \to 0^+} y(r) = \infty$ and $\lim_{r \to \infty} y(r) = 0$. We also prove that the "limiting solution" of this family is the ground state. Solutions of Eq. (1) are radially symmetric standing waves of the nonlinear Schrodinger equation. A central component of our investigation is the associated integral equation

$$(2) \qquad y_k(r) = k\frac{e^{-r}}{r} - \int_r^\infty \frac{t}{r} \sinh(t-r) y_k(t)|y_k(t)|^{p-1} dt, \ \ 0 < r < \infty.$$

Let $2 < p \leq 3$. For each $k > 0$ there is a unique solution $y_k(r)$ of (2), it satisfies (1), and $y_k(r) \sim k\frac{e^{-r}}{r}$ as $r \to \infty$. We make use of (2) to prove

(I) the existence of a bounded interval $(0, k^*)$ such that for each $k \in (0, k^*)$, $y_k'(r) < 0 \ \ \forall r > 0$, $\ y(r) \to \infty$ as $r \to 0^+$, and $y_k(r) \sim k\frac{e^{-r}}{r}$ as $r \to \infty$,

(II) when $k = k^*$, $\ y_{k^*}(r)$ is a ground state solution satisfying $0 < y_{k^*}(0) < \infty$, $y_{k^*}'(0) = 0$, $\ y_{k^*}'(r) < 0 \ \ \forall r > 0$, and $y_k(r) \sim k\frac{e^{-r}}{r}$ as $r \to \infty$.

We also state related open problems for future research.

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1 Introduction

The nonlinear Schrodinger (NLS) equation

$$\mathrm{i}\psi_t + \Delta\psi + \psi|\psi|^{p-1} = 0, \quad p > 1 \tag{1.1}$$

is a canonical equation which arises in studies of laser beam self-focusing, in plasma physics, and in the formation of Bose-Einstein condensates [7, 12, 20]. Coz [6] states "in the theory of nonlinear Schrodinger equations, either solutions spread out or they concentrate at one or more points." In particular, the formation of singular solutions, and peak shaped solutions has played a central role in understanding optical self-focusing, and Bose-Einstein condensate formation. Coz [6] used variational techniques to prove existence of of standing wave solutions, i.e. solutions of (1.1) which have the form $\psi = e^{it}\phi$, where ϕ satisfies

$$\Delta\phi + \phi|\phi|^{p-1} - \phi = 0. \tag{1.2}$$

Subsequently, Byeon and Wang [2] investigated standing wave solutions when a singular potential term is added onto (1.1).

Goal. We focus on the range $2 < p \leq 3$, and the physically relevant dimension $N = 3$, and investigate radially symmetric solutions $\phi = y(r)$, of (1.2), where y satisfies

$$y'' + \frac{2}{r}y' + y|y|^{p-1} - y = 0, \quad 0 < r < \infty. \tag{1.3}$$

Our primary goal is to prove existence, and asymptotic behavior as $r \to \infty$, of a family of singular solutions of (1.3) which satisfy (see Figure 1, right panel)

$$y(r) > 0 \;\; \forall r > 0, \;\; \lim_{r\to 0^+} y(r) = \infty \;\; \text{and} \;\; \lim_{r\to\infty} y(r) = 0. \tag{1.4}$$

Section 3 describes future research into the full regime $N > 1, \; 1 < p < \frac{N+2}{N-2}$.

Previous Results. Before stating our main result (Theorem 1.2 below) we summarize results of previous studies. First, a bound state is a solution of (1.3) that satisfies

$$y(0) = y_0 \neq 0, \;\; y'(0) = 0, \tag{1.5}$$

$$\lim_{r\to\infty}(y(r), y'(r)) = (0, 0). \tag{1.6}$$

A ground state satisfies (1.3), (1.5), (1.6), and $|y(r)| > 0\, \forall r > 0$. Methods used to prove existence of bound states include **(i)** a variational approach and **(ii)** ODE methods.

(i) The Variational Approach. In 1963 Nehari [16] restricted his attention to the physically relevant dimension $N = 3$, and used a variational method to prove His investigation of (1.3) was motivated by studies of spinors [8], nucleons [15], and the nuclear core [21]. Thus, Nehari investigated the problem

$$J = \int_0^\infty \left(y^2(t) + (y'(t))^2\right) dt = \text{minimum}, \tag{1.7}$$

subject to the normalizing condition

$$\int_0^\infty \frac{y^{p+1}(t)}{t^{p-1}} dt = 1, \tag{1.8}$$

where $y(r) \geq 0$ is continuous, $y'(r)$ is piecewise continuous, $y'(0) = 0$, and the integrals (1.7), (1.8) exist. Nehari proved that in this class of functions there is at least one that minimizes J, and it is a ground state (Figure 1). Nehari did not investigate singular solutions. Subsequently, Strauss [19] and Berestycki and Lions [1] used variational methods to prove existence of infinitely many radially symmetric solutions.

(ii) ODE Methods. A standard technique to determine qualitative behavior of solutions of (1.3) is to investigate initial value problem (1.3), (1.5). This approach, combined with energy estimates and comparison techniques has been used to prove existence and uniqueness of ground states and sign changing bound states [4, 5, 9, 11, 13, 14, 18]. A key observation is that the energy functional

$$E = \frac{y'^2}{2} + \frac{|y|^{p+1}}{p+1} - \frac{y^2}{2}, \tag{1.9}$$

satisfies

$$E(0) = \frac{|y_0|^{p+1}}{p+1} - \frac{y_0^2}{2}, \quad \text{and} \quad E' = -\frac{2y'^2}{r} \leq 0 \quad \forall r > 0. \tag{1.10}$$

An important consequence of (1.9)-(1.10) is the following property:

(I) For each $y_0 \neq 0$ the solution of (1.3), (1.5) is bounded on $[0, \infty)$.

Statement Of Main Results. Property **(I)** implies that the existence of unbounded singular solutions satisfying (1.3)-(1.4) cannot be proved by analyzing initial value problem (1.3), (1.5). Also, as pointed out above, Nehari did not prove existence of singular solutions. Thus, to prove the existence of solutions of (1.3)-(1.4) we employ an entirely different approach, which we refer to as "topological shooting from infinity." A central component of this approach is the associated integral equation

$$y_k(r) = k\frac{e^{-r}}{r} - \int_r^\infty \frac{t}{r}\sinh(t-r)y_k(t)|y_k(t)|^{p-1}dt. \tag{1.11}$$

Lemma 1.1 *Let $2 < p \leq 3$ be fixed. For each $k > 0$ there is a unique solution, $y_k(r)$, of (1.11), it satisfies ODE (1.3), and $y_k(r) \sim k\frac{e^{-r}}{r}$ as $r \to \infty$. For each $r > 0$ where $y_k(r)$ continues to exist the solution and its derivatives are continuous functions of r and k.*

Equation (1.11) can be solved (uniquely) by the method of successive approximations, and this technique gives both the solution and its continuous dependence on k. This technique was used by Hastings and McLeod ([9] and [10], pp. 48-49) in proving existence of a unique, monotonically decreasing solution of the Painleve type equation

$$y'' = xy + 2y|y|^\alpha, \quad \alpha > 0, \tag{1.12}$$

$$y(x) \sim \left(-\frac{1}{2}x\right)^{1/\alpha} \quad \text{as} \quad x \to -\infty, \quad \text{and} \quad y(x) \sim Ai(x) \quad \text{as} \quad x \to \infty, \tag{1.13}$$

where $Ai(x)$ is the Airy function. Thus, although the method of proof of Lemma 1.1 is standard, we give a brief sketch of the details in the Appendix.

We now define the topological shooting set

$$S = \left\{ k > 0 \;\middle|\; \text{if } 0 < \hat{k} < k \text{ then } y'_{\hat{k}}(r) < 0 \;\; \forall r > 0 \;\; \text{and} \;\; \lim_{r \to 0^+} y_{\hat{k}}(r) = \infty \right\} \tag{1.14}$$

Theorem 1.2 *Let $2 < p \leq 3$. There exists $k^* > 0$ such that $S = (0, k^*)$.*

(i) Singular Solutions *Let $k \in (0, k^*)$. Then*

$$y_k(r) > 0 \quad \text{and} \quad y'_k(r) < 0 \quad \forall r > 0, \tag{1.15}$$

$$\lim_{r \to 0^+} y_k(r) = \infty \quad \text{and} \quad y_k(r) \sim \; k\frac{e^{-r}}{r} \quad \text{as} \quad r \to \infty. \tag{1.16}$$

(ii) Ground State *Let $k = k^*$. There exists $y^* > 0$ such that*

$$y_{k^*}(r) > 0 \quad \text{and} \quad y'_{k^*}(r) < 0 \quad \forall r > 0, \tag{1.17}$$

$$\lim_{r\to 0^+} (y_{k^*}(r), y'_{k^*}(r)) = (y^*, 0) \quad \text{and} \quad y_k(r) \sim \; k\frac{e^{-r}}{r} \quad \text{as} \quad r \to \infty. \tag{1.18}$$

Remarks. Figure 1 and Figure 2 illustrate the stark difference between solutions of initial value problem (1.3), (1.5) and solutions of the equivalent integral equation (1.11). By combining the results of these diverse analytical approaches, i.e. **(1)** the investigation of behavior of solutions of initial value problem (1.3), (1.5) as r *increases* from $r = 0$, and **(2)** the investigation of behavior of solutions of integral equation (1.11) as r *decreases* from large values, we have obtained new insights into the global structure of radially symmetric standing wave solutions of the nonlinear Schrodinger equation (1.1). Section 2 gives the proof of Theorem 1.2. Section 3 describes open problems, and how a natural "emergemce" phenomenon is observed when r decreases from large values.

2 Proof Of Theorem 1.2.

Throughout the proof we focus on the largest non-negative intervals where solutions of (1.11) are positive. Thus, let $p \in (2, 3]$ be fixed, and for each $k > 0$ define

$$a_k = \inf\{\hat{r} > 0 \mid y_k(r) > 0 \;\; \forall r > \hat{r}\}. \tag{2.1}$$

It follows from (2.1) that (a_k, ∞) is the largest positive interval where $y_k(r) > 0$.

Next, we give an outline of our two part procedure to complete the proof.

(I) First, we prove three auxiliary lemmas.

(a) Lemma 2.1 gives practical upper and lower bounds on solutions y_k of (1.11).

(b) Lemma 2.2 gives a criterion which guarantees that $a_k = 0$, i.e. that $(a_k, \infty) = (0, \infty)$.

(c) Lemma 2.3 gives a criterion for a solution to satisfy $\lim_{r\to 0^+} y_k(r) = \infty$.

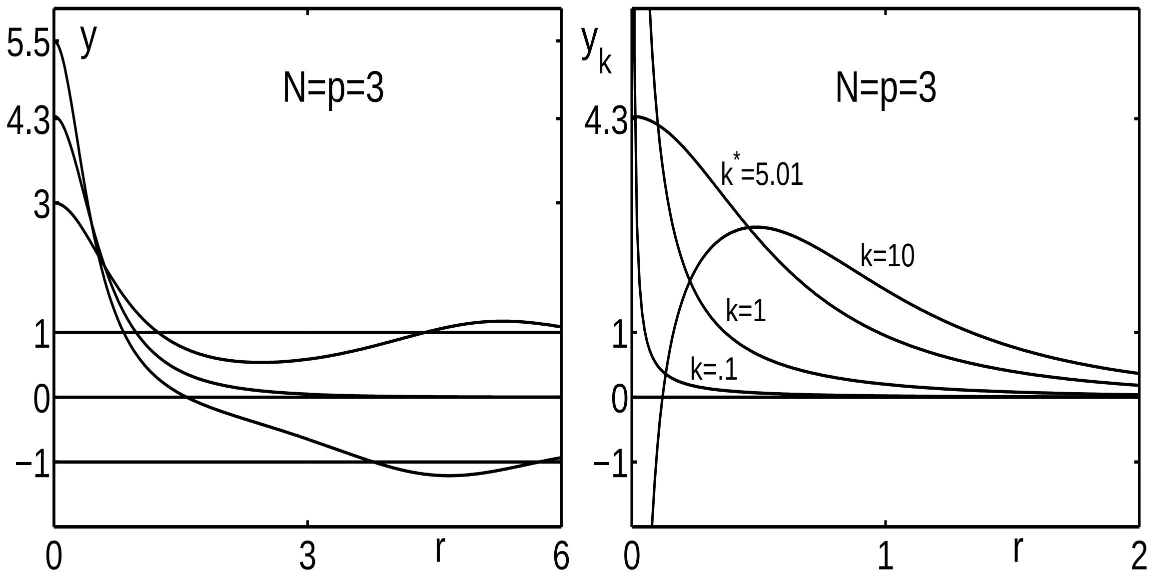

Figure 1: **(Left)** Solutions of initial value problem (1.3), (1.5) when $N = p = 3$; $y_0 = 3$ gives a solution such that $\lim_{r\to 0^+} y(r) = 1$, $y_0 = 4.3$ gives the ground state, $y_0 = 5.5$ gives a solution such that $\lim_{r\to 0^+} y(r) = -1$. **(Right)** Solutions of integral equation (1.11) when $N = p = 3$; $k = .1,\ k = 1$ give positive singular solutions which satisfy **(i)** in Theorem 1.2, $k^* = 5$ gives the ground state which satisfies **(ii)** in Theorem 1.2, and $k = 10$ gives a sign changing singular solution such that $\lim_{r\to 0^+} y_{10}(r) = -\infty$.

(II) We use Lemma 2.1, Lemma 2.2 and Lemma 2.3 to prove the following four lemmas.

(d) Lemma 2.4 shows that $S \neq \phi$.

(e) Lemma 2.5 and Lemma 2.6 show that S is open.

(f) Lemma 2.7 shows that $k_0 \notin S$ for aome $k_0 > 0$.

It follows from Lemma 1.1, the definition of S given in (1.14), and **(d)-(e)-(f)** above that S is a finite, connected, open interval of the form $S = (0, k^*)$, and that condition **(i)** in Theorem 1.2 is satisfied. Finally, to prove that condition **(ii)** holds we show, in Lemma 2.8, that $y_{k^*}(r)$ is a ground state solution.

Part (I) In Lemma 2.3 we derive upper and lower bounds on $y_k(r)$ and $y_k'(r)$ when $r \in (a_k, \infty)$. The proof uses the fact that (1.11) can be written as

$$y_k(r) = k\frac{e^{-r}}{r} - \frac{1}{2}\int_r^\infty \frac{t}{r}\left(e^{t-r} - e^{-(t-r)}\right) y_k(t)|y_k(t)|^{p-1}dt. \tag{2.2}$$

Lemma 2.1 *Let* $2 < p \leq 3$ *and* $k > 0$*. Then, for all* $r \in (a_k, \infty)$*,*

$$k\frac{e^{-r}}{r}\left(1 - \frac{k^{p-1}e^{-(p-1)r}}{(p^2-1)r^{p-1}}\right) \leq y_k(r) \leq k\frac{e^{-r}}{r}, \tag{2.3}$$

$$-k\frac{e^{-r}}{r}\left(1+\frac{1}{r}\right) \leq y_k'(r) \leq -k\frac{e^{-r}}{r}\left(1 + \frac{1}{r} - \frac{pk^{p-1}e^{-(p-1)r}}{(p^2-1)r^{p-1}} - \frac{k^{p-1}e^{-(p-1)r}}{(p^2-1)r^p}\right). \tag{2.4}$$

Proof. First, since $\sinh(t-r) \geq 0$ when $t \geq r$, it follows from (2.2) that

$$y_k(r) \leq k\frac{e^{-r}}{r} \quad \forall r \in (a_k, \infty). \tag{2.5}$$

Next, we substitute $y_k(r) \leq k\frac{e^{-r}}{r}$ into (2.2) and obtain

$$y_k(r) \geq k\frac{e^{-r}}{r} - \frac{1}{2}\int_r^\infty \frac{t}{r}\left(e^{t-r} - e^{-(t-r)}\right) k^p\frac{e^{-pt}}{t^p}dt \quad \forall r \in (a_k, \infty). \tag{2.6}$$

Because $\frac{1}{t} \leq \frac{1}{r}$ when $t \geq r$, (2.6) reduces to

$$y_k(r) \geq k\frac{e^{-r}}{r} - \frac{k^p}{2r^p}\int_r^\infty \left(e^{t-r} - e^{-(t-r)}\right) e^{-pt}dt \quad \forall r \in (a_k, \infty). \tag{2.7}$$

An evaluation of the right side of (2.7) gives

$$y_k(r) \geq k\frac{e^{-r}}{r}\left(1 - \frac{k^{p-1}e^{-(p-1)r}}{(p^2-1)r^{p-1}}\right) \quad \forall r \in (a_k, \infty). \tag{2.8}$$

Next, we prove (2.4). Differentiating both sides of (1.11) with respect to r gives

$$y_k'(r) = -k\frac{e^{-r}}{r} - k\frac{e^{-r}}{r^2} + \int_r^\infty t\left(\frac{\cosh(t-r)}{r} + \frac{\sinh(t-r)}{r^2}\right) y_k(t)|y_k(t)|^{p-1}dt \tag{2.9}$$

for $r > a_k$. Substituting $y_k(r) \leq k\frac{e^{-r}}{r}$ into (2.9) and using $\frac{1}{t} \leq \frac{1}{r}$, we obtain

$$y_k'(r) \leq -k\frac{e^{-r}}{r} - k\frac{e^{-r}}{r^2} + k^p\int_r^\infty \left(\frac{\cosh(t-r)}{r^p} + \frac{\sinh(t-r)}{r^{p+1}}\right) e^{-pt}dt \tag{2.10}$$

for all $r \in (a_k, \infty)$. An evaluation of the right side of (2.10) gives

$$y_k'(r) \leq -k\frac{e^{-r}}{r}\left(1 + \frac{1}{r} - \frac{pk^{p-1}e^{-(p-1)r}}{(p^2-1)r^{p-1}} - \frac{k^{p-1}e^{-(p-1)r}}{(p^2-1)r^p}\right) \quad \forall r \in (a_k, \infty). \tag{2.11}$$

From (2.9) we get $y_k(r) \geq -k\frac{e^{-r}}{r}\left(1 + \frac{1}{r}\right) \quad \forall r \in (a_k, \infty)$. This completes the proof.

In Lemma 2.2 we use Lemma 2.1 to prove a criterion which gurantees that $a_k = 0$.

Lemma 2.2 *Let* $2 < p \leq 3$ *and* $k > 0$. *If*

$$y_k(r) > 0 \quad \text{and} \quad y_k'(r) < 0 \quad \forall r \in (a_k, \infty), \tag{2.12}$$

then $a_k = 0$.

Proof. Suppose, however, that $p \in (2, 3]$ and $k > 0$ exist such that (2.12) holds, and $a_k > 0$. Then (2.1) and (2.12) imply that $y_k(r)$ ceases to exist as $r \to a_k^+$, hence

$$\lim_{r \to a_k^+} y_k(r) = \infty. \tag{2.13}$$

We conclude from (2.1), (2.3) and (2.12) that

$$0 < y_k(r) \leq k\frac{e^{-r}}{r} \quad \forall r \in (a_k, \infty). \tag{2.14}$$

It follows from (2.14) that $\lim_{r \to a_k^+} y_k(r) \leq k\frac{e^{-a_k}}{a_k} < \infty$, contradictiing (2.13). Thus, we conclude that $a_k = 0$, and the lemma is proved.

Next, in Lemma 2.3 we derive a criterion which guarantees that a positive solution of (1.11) satisfies $y_k(r) \to \infty$ as $r \to 0^+$. The proof makes use of the function

$$H_k(r) = r y_k(r), \quad r > a_k, \tag{2.15}$$

which satisfies

$$H_k'(r) = r y_k'(r) + y_k(r) \quad \text{and} \quad H_k''(r) = H_k(1 - y_k^{p-1}(r)), \quad r > a_k. \tag{2.16}$$

Lemma 2.3 **(i)** *Let $2 < p < 3$ and $k > 0$. Assume that $\bar{r} \in \left(a_k, \frac{3-p}{p-1}\right)$ exists such that*

$$y_k(\bar{r}) > 1,\ y_k'(\bar{r}) < 0,\ y_k''(\bar{r}) > 0 \ \text{ and } \ k^{p-1}\bar{r}^{3-p}e^{-(p-1)\bar{r}} \leq \frac{2}{p}. \tag{2.17}$$

Then $a_k = 0$,

$$y_k(r) > 0,\ y_k'(r) < 0 \ \text{ and } \ y_k''(r) > 0 \ \ \forall r \in (0, \bar{r}), \quad \text{and} \quad \lim_{r \to 0^+} y_k(r) = \infty. \tag{2.18}$$

(ii) *Let $p = 3$ and $k > 0$. Assume that $\bar{r} > 0$ exists such that*

$$y_k(\bar{r}) > 1,\ y_k'(\bar{r}) < 0,\ y_k''(\bar{r}) > 0,\ 0 < H_k(\bar{r}) < \sqrt{\frac{2}{3}} \ \text{ and } \ H_k'(\bar{r}) > 0. \tag{2.19}$$

Then $a_k = 0$ and property (2.18) holds

Proof. Let $2 < p \leq 3$ and $k > 0$. A differentiation of (1.3) show that y_k satisfies

$$y_k''' + \frac{2}{r} y_k'' = \frac{p}{r^2}\left(\frac{2}{p} - r^2 y_k^{p-1}\right) y_k' + y_k', \quad r > a_k. \tag{2.20}$$

We will also make use of the fact that (2.20) can be written as

$$(r^2 y_k'')' = p\left(\frac{2}{p} - r^2 y_k^{p-1}\right) y_k' + r^2 y_k', \quad r > a_k. \tag{2.21}$$

Suppose, for contradiction that there is an $\hat{r} \in (a_k, \bar{r})$ such that

$$y_k''(r) > 0 \ \ \forall r \in (\hat{r}, \bar{r}) \ \text{ and } \ y_k''(\hat{r}) = 0. \tag{2.22}$$

Then

$$y_k'''(\hat{r}) \geq 0. \tag{2.23}$$

It follows from (2.22) that

$$y_k'(r) < 0 \ \text{ and } \ y_k(r) > 1 \ \ \forall r \in [\hat{r}, \bar{r}). \tag{2.24}$$

To obtain a contradiction to (2.23) we need an upper bound on the term $r^2 y_k^{p-1}$ which appears in (2.20). For this we consider two cases, (i) $2 < p < 3$ and (ii) $p = 3$.

(i) Let $2 < p < 3$. It follows from (2.3) that

$$r^2 y_k^{p-1}(r) \leq k^{p-1} r^{3-p} e^{-(p-1)r} \ \ \forall r \in (a_k, \infty). \tag{2.25}$$

It is easily verified that $\left(r^{3-p}e^{-(p-1)r}\right)' > 0$ when $r \in \left(a_k, \frac{3-p}{p-1}\right)$. This property, our assumption that $\hat{r} \in (a_k, \bar{r})$, assumption (2.17) and (2.25) imply that

$$\hat{r}^2 y_k^{p-1}(\hat{r}) \leq \hat{r}^{3-p} e^{-(p-1)\hat{r}} < \frac{2}{p}. \tag{2.26}$$

From (2.24) we get $y_k'(\hat{r}) < 0$. This fact, (2.20) and (2.26) imply that $y_k'''(\hat{r}) < 0$, contradicting (2.23).

(ii) Let $p = 3$. It follows from (2.16) and (2.24) that $H_k = r y_k$ satisfies $H_k''(r) < 0 \ \forall r \in [\hat{r}, \bar{r})$. This property and hypothesis (2.19) imply that

$$H_k'(r) > 0 \ \text{ and } \ 0 < H_k^2(r) < \frac{2}{3} \ \ \forall r \in [\hat{r}, \bar{r}). \tag{2.27}$$

Thus, $0 < H_k^2(\hat{r}) = \hat{r}^2 y_k^2(\hat{r}) < \frac{2}{3}$ and we conclude from (2.20) and (2.24) that $y_k'''(\hat{r}) < 0$, again contradicting (2.23). Thus, for each $p \in (2, 3]$, we conclude that $y_k''(r) > 0$ for all $r \in (a_k, \bar{r})$. From this it follows that $y_k'(r) < 0$ and $1 < y_k(r) < k\frac{e^{-r}}{r}$ for all $r \in (a_k, \bar{r})$. These bounds and Lemma 2.2 imply that $a_k = 0$, hence

$$y_k''(r) > 0, \ y_k'(r) < 0 \ \text{ and } \ 1 < y_k(r) < k\frac{e^{-r}}{r} \ \ \forall r \in (0, \bar{r}). \tag{2.28}$$

It remains to prove that $\lim_{r\to 0^+} y_k(r) = \infty$. Suppose, for contradiction, that $p \in (2, 3]$ and $M_1 > 1$ exist such that

$$1 < y_k(r) \leq M_1 \ \ \forall r \in (0, \bar{r}). \tag{2.29}$$

Let $y_k'(\bar{r}) = -M_2$. Since $y_k''(r) > 0 \ \ \forall r \in (0, \bar{r})$, we conclude that

$$y_k'(r) < -M_2 \ \ \forall r \in (0, \bar{r}]. \tag{2.30}$$

Let $r_1 \in (0, \bar{r}]$ satisfy

$$-\frac{M_2}{2} + \frac{M_1^p r_1}{3} < 0. \tag{2.31}$$

From (1.3) and (2.29) we get

$$\left(r^2 y_k'(r)\right)' \geq -r^2 M_1^p \quad \forall r \in (0, \bar{r}]. \tag{2.32}$$

Integrating (2.32) from r to r_1, and using (2.30)-(2.31), we obtain

$$r^2 y_k'(r) \leq r_1^2 \left(y_k'(r_1) + \frac{M_1^p r_1}{3}\right) \leq -\frac{{r_1}^2 M_2}{2} \quad \forall r \in (0, r_1]. \tag{2.33}$$

Dividing both sides of (2.33) by r^2, and integrating from r to r_1 gives

$$y_k(r) \geq y_k(r_1) + \frac{\bar{r}^2 M_2}{2}\left(\frac{1}{r} - \frac{1}{r_1}\right) > M_1 \quad \text{when} \quad 0 < r \ll r_1, \tag{2.34}$$

contradicting (2.29). This completes the proof.

Part (II) In this part we prove Lemma 2.4, Lemma 2.5, Lemma 2.6 and Lemma 2.7. First, in Lemma 2.4 we prove that $S \neq \phi$.

Lemma 2.4 **(i)** *Let* $p \in (2,3)$. *Then* $\left(0, \left(\frac{2}{p}\left(\frac{3-p}{p-1}\right)^{p-3} e^{3-p}\right)^{\frac{1}{p-1}}\right) \subset S$.
(ii) *Let* $p = 3$. *Then* $\left(0, \sqrt{\frac{2}{3}}\right) \subset S$.

Proof. **(i)** Let $p \in (2,3)$. The key to the proof is to obtain an upper bound on the term $r^2 y_k^{p-1}$ which appears on the right side of (2.20). This bound ((2.38) below) allows us to determine the sign of $y''(r)$. It follows from (2.3) that

$$r^2 y_k^{p-1} \leq k^{p-1} r^{3-p} e^{-(p-1)r} \quad \forall r > a_k. \tag{2.35}$$

We need an upper bound on the function

$$f(r) = r^{3-p} e^{-(p-1)r}, \; r \geq 0. \tag{2.36}$$

It is easily verified that f attains a positive maximum at $r_p = \frac{3-p}{p-1}$, and that

$$0 \leq f(r) \leq f(r_p) \leq \left(\frac{3-p}{p-1}\right)^{3-p} e^{p-3} \quad \forall r \geq 0. \tag{2.37}$$

Combining (2.35), (2.36) and (2.37), we obtain

$$0 < r^2 y_k^{p-1}(r) \le k^{p-1} r^{3-p} e^{-(p-1)r} \le k^{p-1} \left(\frac{3-p}{p-1}\right)^{3-p} e^{p-3} < \frac{2}{p} \;\; \forall r > a_k \tag{2.38}$$

if $0 < k < \left(\frac{2}{p}\left(\frac{3-p}{p-1}\right)^{p-3} e^{3-p}\right)^{\frac{1}{p-1}}$. Next, we conclude from (1.3), (2.3), (2.4) that

$$0 < y_k(r) < 1, \; y_k'(r) < 0 \;\text{ and }\; y_k''(r) > 0 \;\text{ when }\; r \gg 1. \tag{2.39}$$

Combining (2.21) with (2.38) and (2.39), we conclude that

$$\left(r^2 y_k''(r)\right)' < 0 \;\text{ and }\; y_k''(r) > 0 \;\; \forall r \in (a_k, \infty). \tag{2.40}$$

It follows from (2.3), (2.39), (2.40) that if $0 < k < \left(\frac{2}{p}\left(\frac{3-p}{p-1}\right)^{p-3} e^{3-p}\right)^{\frac{1}{p-1}}$, then

$$0 < y_k(r) \le k\frac{e^{-r}}{r}, \; y_k'(r) < 0 \;\text{ and }\; y_k''(r) > 0 \;\; \forall r \in (a_k, \infty). \tag{2.41}$$

We conclude from Lemma 2.2 and (2.41) that

$$a_k = 0 \;\; \forall k \in \left(0, \left(\frac{2}{p}\left(\frac{3-p}{p-1}\right)^{p-3} e^{3-p}\right)^{\frac{1}{p-1}}\right). \tag{2.42}$$

Thus, the bounds in (2.41) hold for all $r > 0$. It remains to prove that

$$\lim_{r\to 0^+} y_k(r) = \infty \;\; \forall k \in \left(0, \left(\frac{2}{p}\left(\frac{3-p}{p-1}\right)^{p-3} e^{3-p}\right)^{\frac{1}{p-1}}\right). \tag{2.43}$$

Suppose, however, that $k \in \left(0, \left(\frac{2}{p}\left(\frac{3-p}{p-1}\right)^{p-3} e^{3-p}\right)^{\frac{1}{p-1}}\right)$ and $M_1 > 2$ exist such that

$$0 < (y_k(r))^p \le M_1 \;\; \forall r > 0. \tag{2.44}$$

The proof of **(i)** is complete if we obtain a contradiction to (2.44). For this we focus on $(0, 1]$ and let $L = -y_k'(1)$. Let $r_1 \in (0, 1]$ satisfy

$$-\frac{L}{2} + \frac{M_1 r_1}{3} < 0. \tag{2.45}$$

It follows from (1.3), and the fact that $y_k(r) > 0 \;\; \forall r > 0$, that $(r^2 y_k')' \ge -r^2 M_1 \;\; \forall r \in (0, r_1]$. Integrating from r to r_1, we obtain

$$r^2 y_k'(r) \le r_1^2 y_k'(r_1) + \frac{M_1}{3}\left(r_1^3 - r^3\right) \le r_1^2 y_k'(r_1) + \frac{M_1}{3} r_1^3 \;\; \forall r \in (0, r_1]. \tag{2.46}$$

Next, observe that $y_k'(r_1) < -L = y_k'(1)$ since (2.41) gives $y_k''(r) > 0$ for all $r > 0$. Combining this inequality with (2.45) and (2.46), we obtain $y_k'(r) \le -\frac{Lr_1^2}{2r^2} \quad \forall r \in (0, r_1]$. Integrating from r to r_1 gives

$$y_k(r) \ge y_k(r_1) + \frac{Lr_1^2}{2}\left(\frac{1}{r} - \frac{1}{r_1}\right) > (M_1)^{1/p} \quad \text{when} \quad 0 < r \ll r_1, \tag{2.47}$$

contradicting (2.44).

(ii) Let $p = 3$ and let $k \in \left(0, \sqrt{\frac{2}{3}}\right)$. When $p = 3$, equation (2.21) becomes

$$(r^2 y_k'')' = \frac{3}{r^2}\left(\frac{2}{3} - r^2 y_k^2\right) y_k' + r^2 y_k', \quad r > a_k. \tag{2.48}$$

As above, we need to show that the term $\frac{2}{3} - r^2 y_k^2 > 0 \ \ \forall r \in (a_k, \infty)$. It follows from upper bound (2.3), and the supposition that $k \in \left(0, \sqrt{\frac{2}{3}}\right)$, that

$$r^2 y_k^2 < k^2 e^{-2r} < \frac{2}{3} \ \ \forall r > a_k. \tag{2.49}$$

Also, as above, we conclude from (1.3), (2.3), (2.4) that

$$0 < y_k(r) < 1, \ y_k'(r) < 0 \ \text{ and } \ y_k''(r) > 0 \ \text{ when } \ r \gg 1. \tag{2.50}$$

Combining (2.48) with (2.49) and (2.50), we conclude that

$$\left(r^2 y_k''(r)\right)' < 0 \ \text{ and } \ y_k''(r) > 0 \ \ \forall r \in (a_k, \infty). \tag{2.51}$$

It follows from (2.3), (2.50) and (2.51) that if $k \in \left(0, \sqrt{\frac{2}{3}}\right)$, then

$$0 < y_k(r) \le k\frac{e^{-r}}{r}, \ y_k'(r) < 0 \ \text{ and } \ y_k''(r) > 0 \ \ \forall r \in (a_k, \infty). \tag{2.52}$$

We conclude from Lemma 2.2 and (2.52) that $a_k = 0$. Thus, the bounds in (2.52) hold for all $r > 0$. It remains to prove that

$$\lim_{r \to 0^+} y_k(r) = \infty \ \text{ when } \ k \in \left(0, \sqrt{\frac{2}{3}}\right). \tag{2.53}$$

Suppose, however, that $k \in \left(0, \sqrt{\frac{2}{3}}\right)$ and $M_1 > 2$ exist such that

$$0 < (y_k(r))^3 \le M_1 \ \ \forall r > 0. \tag{2.54}$$

The proof of **(i)** is complete if we obtain a contradiction to (2.54). Such contradiction is obtained exactly as in part **(i)**, and we omit the details for the sake of brevity.

Our next goal is to prove that S is open (Lemma 2.6). For this we first need to prove the following auxiliary result.

Lemma 2.5 **(i)** *Let* $2 < p < 3$ *and* $k \in S$. *Then*

$$r^2 y_k^{p-1}(r) \to 0 \quad \text{as} \quad r \to 0^+. \tag{2.55}$$

(ii) *Let* $p = 3$ *and* $k \in S$. *Then*

$$r y_k(r) \to 0 \quad \text{as} \quad r \to 0^+. \tag{2.56}$$

Proof. **(i)** Let $2 < p < 3$ and $k \in S$. It follows from (2.3) that

$$0 < r^2 y_k^{p-1}(r) \le k^{p-1} r^{3-p} e^{-(p-1)r} \quad \forall r \in (0, \infty). \tag{2.57}$$

Since $2 < p < 3$, we observe that $r^{3-p}e^{-(p-1)r} \to 0$ as $r \to 0^+$. Combining this property with (2.57) gives property (2.55).

(ii) Let $p = 3$ and $k \in S$. Since $k \in S$, there is an $\bar{r} > 0$ such that

$$y_k(r) > 1, \ y_k'(r) < 0 \ \ \forall r \in (0, \bar{r}), \quad \text{and} \quad \lim_{r \to 0^+} y_k(r) = \infty. \tag{2.58}$$

Suppose that (2.56) doesn't hold. Then either $L > 0$ and $r_L \in (0, \bar{r})$ exist such that

$$r y_k(r) \ge L \quad \forall r \in (0, r_L], \tag{2.59}$$

or else

$$0 = \liminf_{r \to 0^+} r y_k(r) < \limsup_{r \to 0^+} r y_k(r). \tag{2.60}$$

First, assume that (2.59) holds. Let $r_L \in (0, \bar{r})$ be chosen so small that

$$2 < \frac{L}{r} \le y_k(r) \quad \forall r \in (0, r_L]. \tag{2.61}$$

it follows from (1.3) and (2.61) that

$$\left(r^2 y_k'\right)' = r^2 \left(y_k - \frac{y_k^3}{2}\right) - r^2 \frac{y_k^3}{2} \le -r^2 \frac{y_k^3}{2} \le -\frac{L^3}{2r} \quad \forall r \in (0, r_L]. \tag{2.62}$$

An integration from r to r_L gives

$$r^2 y_k'(r) \geq r_L^2 y_k'(r_L) + \frac{L^3}{2r}\left(\ln(r_L) - \ln(r)\right) > 0 \text{ when } 0 < r \ll r_L, \tag{2.63}$$

contradictiing (2.58). Thus, (2.59) cannot hold. Finally, suppose that (2.60) holds. Then there exists $\tilde{r} > 0$ such that $y_k(r)$ and $H_k(r) = r y_k(r)$ satisfy

$$y_k(\tilde{r}) > 1, \ H_k'(\tilde{r}) = 0 \text{ and } H_k''(\tilde{r}) \geq 0. \tag{2.64}$$

However, it follows from the fact that $y_k(\tilde{r}) > 1$ and (2.16) that $H_k''(\tilde{r}) = H_k(\tilde{r})(1 - y_k^2(\tilde{r})) < 0$, contradicting (2.64). Thus, (2.60) cannot hold. Therefore, we conclude that property (2.56) holds and the proof is complete.

We make use of Lemma 2.5 to prove

Lemma 2.6 *Let $2 < p \leq 3$. Then S is open.*

Proof. Let $\bar{k} \in S$. The definition of S implies that $\bar{r} > 0$ exists such that

$$y_{\bar{k}}(r) > 1, \ y_{\bar{k}}'(r) < 0 \ \ \forall r \in (0, \bar{r}], \text{ and } \lim_{r \to 0^+} y_{\bar{k}}(r) = \infty. \tag{2.65}$$

To complete the proof we consider two separate cases, **(i)** $2 < p < 3$ and **(ii)** $p = 3$.

(i) $2 < p < 3$. It follows from criterion (2.17) in Lemma 2.3 that we only need to prove

$$y_{\bar{k}}(\hat{r}) > 1, \ y_{\bar{k}}'(\hat{r}) < 0, \ y_{\bar{k}}''(\hat{r}) > 0 \text{ and } k^{p-1}\hat{r}^{3-p}e^{-(p-1)\hat{r}} < \frac{2}{p}, \tag{2.66}$$

for some $\hat{r} \in \left(0, \frac{3-p}{p-1}\right)$ when $0 < |k - \bar{k}| \ll 1$. Thus, let $\bar{r}$ be chosen to also satisfy

$$\bar{r} \in \left(0, \frac{3-p}{p-1}\right) \text{ and } \bar{k}^{p-1}\bar{r}^{3-p}e^{-(p-1)\bar{r}} < \frac{2}{p}. \tag{2.67}$$

Observe that $\left(r^{3-p}e^{-(p-1)r}\right)' > 0 \ \ \forall r \in \left(0, \frac{3-p}{p-1}\right)$. This, (2.66) and (2.67) imply that

$$r^2 y_{\bar{k}}^{p-1}(r) \leq \bar{k}^{p-1}\bar{r}^{p-1}e^{-(p-1)\bar{r}} < \frac{2}{p} \ \ \forall r \in (0, \bar{r}]. \tag{2.68}$$

Finally, since $y_{\bar{k}}(r) \to \infty$ as $r \to 0^+$, there is an $\hat{r} \in (0, \bar{r})$ such that $y_{\bar{k}}''(\hat{r}) > 0$. Otherwise, if $y_{\bar{k}}''(r) < 0$ for all $r \in (0, \bar{r}]$, then two integrations from r to $\bar{r}$ give

$$y_{\bar{k}}(r) \leq y_{\bar{k}}(\bar{r}) + y_{\bar{k}}'(\bar{r})(r - \bar{r}) < \infty \ \ \forall r \in [0, \bar{r}], \tag{2.69}$$

contradicting (2.65). Thus, (2.66) holds when $k = \bar{k}$. It follows from continuity that (2.66) holds when $0 <\leq |k - \bar{k}| \ll 1$. Thus, criterion (2.17) for blowup of solutions in Lemma 2.17 is satisfied when $0 \leq |k - \bar{k}| \ll 1$. This completes the proof of (i)

(ii) $p = 3$. By Lemma 2.3, we only need to prove that criterion (2.19) holds for some $\bar{r} > 0$ when $0 \leq |k - \bar{k}| \ll 1$. It follows from (2.16), (2.58), (2.65) that $H_{\bar{k}} = r y_{\bar{k}}$ satisfies

$$H_{\bar{k}}(r) > 0, \ H''_{\bar{k}}(r) < 0 \ \ \forall r \in (0, \bar{r}], \quad \text{and} \quad \lim_{r \to 0} H_{\bar{k}}(r) = 0. \tag{2.70}$$

We conclude from (2.70) that $\tilde{r}$ can be chosen to satisfy the additional property

$$0 < H_{\bar{k}}(\bar{r}) < \sqrt{\frac{2}{3}}, \quad \text{and} \quad H'_{\bar{k}}(r) > 0 \ \forall r \in (0, \bar{r}]. \tag{2.71}$$

Next, we claim that there is an $\hat{r} \in (0, \bar{r}]$ such that

$$y''_{\bar{k}}(\hat{r}) > 0. \tag{2.72}$$

If not then $y''_{\bar{k}}(r) < 0 \ \ \forall r \in (0, \bar{r}]$, and two integrations give (2.69), again contradicting (2.65). Thus, (2.72) holds at some $\hat{r} \in (0, \bar{r}]$, and we conclude from (2.65), (2.71) that

$$y_{\bar{k}}(\hat{r}) > 1, \ y'_{\bar{k}}(\hat{r}) < 0, \ 0 < H_{\bar{k}}(\hat{r}) < \sqrt{\frac{2}{3}}, \quad \text{and} \quad H'_{\bar{k}}(\hat{r}) > 0. \tag{2.73}$$

Continuity implies that properties (2.72) and (2.73) hold if $0 \leq |k - \bar{k}| \ll 1$. Thus, when $p = 3$, criterion (2.19) for blowup of solutions in Lemma 2.3 is satisfied when $0 \leq |k - \bar{k}| \ll 1$. This completes the proof of (ii).

We have now proved that S is non-empty and open. Finally, we prove

Lemma 2.7 *Let $2 < p \leq 3$. There exists $k_0 > 0$ such that $k_0 \notin \sup S$.*

Proof. It follows from (2.3) that for each $k_0 > 0$, the solution y_{k_0} satisfies

$$k_0 \frac{e^{-r}}{r} \left(1 - \frac{k_0^{p-1} e^{-(p-1)r}}{(p^2 - 1) r^{p-1}} \right) \leq y_{k_0}(r) \leq k_0 \frac{e^{-r}}{r} \quad \forall r \in (a_{k_0}, \infty). \tag{2.74}$$

To proceed further we consider two cases, **(i)** $2 < p < 3$ and **(ii)** $p = 3$.

(i) Let $2 < p < 3$, and define r_0 and corresponding $k_0 > 0$ by

$$r_0 = \sqrt{\frac{128}{79} \left[\frac{8}{7} \left(\frac{8}{p^2 - 1} \right)^{1/(p-1)} - 1 \right] + \frac{2\pi}{\sqrt{p - 1}}}, \quad k_0 = \left(\frac{p^2 - 1}{8} \right)^{1/(p-1)} r_0 e^{r_0} \tag{2.75}$$

Our goal is to prove that $k_0 \notin S$. For this we need upper and lower bounds on $y_{k_0}(r_0)$, and an upper bound on $y'_{k_0}(r_0)$. First, observe that for any $r_0 > 0$,

$$1 - \frac{k_0^{p-1} e^{-(p-1)r}}{(p^2-1)r^{p-1}} = 1 - \frac{1}{8}\left(\frac{r_0}{r} e^{(r_0 - r)}\right)^{(p-1)} \geq \frac{7}{8} \quad \forall r \geq r_0. \tag{2.76}$$

Thus, for each $r_0 > 0$, if $k_0 = \left(\frac{p^2-1}{8}\right)^{1/(p-1)} r_0 e^{r_0}$ then (2.74) and (2.76) give

$$0 < \frac{7}{8}\left(\frac{p^2-1}{8}\right)^{1/(p-1)} \left(\frac{r_0}{r}\right) e^{r_0 - r} \leq y_{k_0}(r) \leq \left(\frac{p^2-1}{8}\right)^{1/(p-1)} \left(\frac{r_0}{r}\right) e^{r_0 - r} < 1 \tag{2.77}$$

for all $r \geq r_0$. From (2.77) we conclude that $r_0 > a_{k_0}$, and

$$0 < \frac{7}{8}\left(\frac{p^2-1}{8}\right)^{1/(p-1)} \leq y_{k_0}(r_0) \leq \left(\frac{p^2-1}{8}\right)^{1/(p-1)} < 1. \tag{2.78}$$

We also need an upper bound on $y'_{k_0}(r_0)$. For this we make use of the functional

$$E = \frac{1}{2}\left(y'_{k_0}\right)^2 + \frac{1}{p+1}\left(y_{k_0}\right)^{p+1} - \frac{1}{2}\left(y_{k_0}\right)^2, \tag{2.79}$$

where $E' = -\frac{2}{r}\left(y'_{k_0}\right)^2 \leq 0 \;\; \forall r > a_{k_0}$ and $E(\infty) = 0$. Thus, $E(r) > 0 \;\; \forall r \geq r_0$ and

$$\frac{1}{2}\left(y'_{k_0}(r_0)\right)^2 > \frac{1}{2}\left(y_{k_0}(r_0)\right)^2 - \frac{1}{p+1}\left(y_{k_0}(r_0)\right)^{p+1}. \tag{2.80}$$

From (2.78) and (2.80), and using the fact that $1 < p \leq 3$, we obtain the upper bound

$$y'_{k_0}(r_0) \leq -\frac{7}{8}\sqrt{\frac{79}{128}}\left(\frac{p^2-1}{8}\right)^{1/(p-1)}. \tag{2.81}$$

We conclude from (1.3), (2.78), (2.81) that $y''_{k_0}(r) = -\frac{2}{r} y'_{k_0}(r) + y_{k_0}(r) - y_{k_0}|y|^{p-1}(r) > 0$ and $y'_{k_0}(r) \leq y'_{k_0}(r_0)$ for $r < r_0$ as long as $y_{k_0}(r) < 1$. Integrating $y'_{k_0}(r) \leq y'_{k_0}(r_0)$ from r to r_0, and making use of (2.78) and (2.81), we conclude that

$$y_{k_0}(r) \geq \frac{7}{8}\left(\frac{p^2-1}{8}\right)^{1/(p-1)} - \frac{7}{8}\sqrt{\frac{79}{128}}\left(\frac{p^2-1}{8}\right)^{1/(p-1)} (r - r_0), \tag{2.82}$$

for $r < r_0$ as long as $y_{k_0}(r) < 1$. Next, observe that

$$\frac{7}{8}\left(\frac{p^2-1}{8}\right)^{1/(p-1)} - \frac{7}{8}\sqrt{\frac{79}{128}}\left(\frac{p^2-1}{8}\right)^{1/(p-1)} (\bar{r} - r_0) = 1 \tag{2.83}$$

when

$$\bar{r} = r_0 + \frac{\frac{7}{8}\left(\frac{p^2-1}{8}\right)^{1/(p-1)} - 1}{\frac{7}{8}\sqrt{\frac{79}{128}}\left(\frac{p^2-1}{8}\right)^{1/(p-1)}}. \tag{2.84}$$

It follows from (2.75) and (2.84) that

$$\bar{r} = \frac{2\pi}{\sqrt{p-1}}. \tag{2.85}$$

We conclude from (2.81), (2.82), (2.83), (2.84) and (2.85) that there is an $r_1 \in \left[\frac{2\pi}{\sqrt{p-1}}, r_0\right)$ such that $y_{k_0}(r_0) \leq y_{k_0}(r) < 1$ for all $r \in (r_1, r_0]$,

$$y_{k_0}(r_1) = 1 \text{ and } y'_{k_0}(r_1) \leq y'_{k_0}(r_0) < 0. \tag{2.86}$$

Next, we determine the behavior of $y_{k_0}(r)$ when $r < r_1$. First, we solve

$$v'' + \frac{2}{r}v' + (p-1)v = 0, \quad v(r_1) = 0 \text{ and } v'(r_1) = y'_{k_0}(r_1). \tag{2.87}$$

There exist unique A and B such that $v(r) = A\frac{\sin(\sqrt{p-1}r)}{r} + B\frac{\cos(\sqrt{p-1}r)}{r}$. Thus, $v(r)$ has period $\frac{2\pi}{\sqrt{p-1}}$, and it follows that $r_2 \in \left[r_1 - \frac{2\pi}{\sqrt{p-1}}, r_1\right)$ exists such that

$$v(r) > 0 \ \forall r \in (r_2, r_1), \ v(r_2) = 0 \text{ and } v'(r_2) > 0. \tag{2.88}$$

We claim that $y_{k_0}(\tilde{r}) = 0$ for some $\tilde{r} \in [r_2, r_1)$. Suppose, however, that

$$y_{k_0}(r) > 1 \ \forall r \in [r_2, r_1). \tag{2.89}$$

It is easily verified that the wronskian $vy'_{k_0} - (y_{k_0} - 1)v'$ satisfies

$$\left(r^2\left(vy'_{k_0} - (y_{k_0}-1)v'\right)\right)' = -r^2 v\left(y_{k_0}^p - y_{k_0} - (p-1)(y_{k_0}-1)\right) < 0 \tag{2.90}$$

for all $r \in (r_2, r_1)$. It follows from (2.86), (2.88), (2.89) and (2.90) that

$$r^2\left(v(r)y'_{k_0}(r) - (y_{k_0}(r)-1)v'(r)\right) > 0 \ \forall r \in [r_2, r_1). \tag{2.91}$$

When $r = r_2$, (2.91) reduces to $(y_{k_0}(r_2) - 1)v'(r_2) < 0$, a contradiction since $v'(r_1) > 0$ and our supposition in (2.89) that $y_{k_0}(r_2) > 1$. Thus, $\tilde{r} \in [r_2, r_1)$ exists with $y_{k_0}(\tilde{r}) = 0$. In turn, since $y_{k_0}(r_0) = 0$ and $y'_{k_0}(r_0) < 0$, we conclude that $\hat{r} \in (\tilde{r}, r_0)$ exists such that $y'_{k_0}(\hat{r}) = 0$. Thus, $k_0 \notin S$, and the proof is complete,

Lemma 2.8 *Let $2 < p \leq 3$ and $k^* = \sup S$. Then $y_{k^*}(r)$ is a ground state solution.*

Proof. We need to prove that

$$y_{k^*}(r) > 0 \text{ and } y'_{k^*}(r) < 0 \ \ \forall r > 0, \ \lim_{r \to 0^+} y_{k^*}(r) < \infty, \ \lim_{r \to 0^+} y'_{k^*}(r) = 0. \tag{2.92}$$

Suppose, first of all, that there exists $\bar{r} > 0$ such that

$$y_{k^*}(r) > 0 \text{ and } y'_{k^*}(r) < 0 \ \ \forall r > \bar{r} \text{ and } y'_{k^*}(\bar{r}) = 0. \tag{2.93}$$

Then $y''_{k^*}(\bar{r}) \leq 0$. If $y''_{k^*}(\bar{r}) = 0$ then (1.3) implies that $y'_{k^*}(\bar{r}) = 0$ and $y_{k^*}(\bar{r}) = 1$, contradicting uniqueness of the constant solution $y \equiv 1$. Thus, $y''_{k^*}(\bar{r}) < 0$. This and continuity imply that if $0 < k^* - k \ll 1$, then $\bar{r}_k > 0$ exists such that $y'_k(r) < 0 \ \ \forall r > \bar{r}_k$ and $y'_k(\bar{r}_k) = 0$. Therefore, $k \notin S$ if $0 \leq k^* - k \ll 1$, contradicting $k^* = \sup S$. Thus,

$$y_{k^*}(r) > 0 \text{ and } y'_{k^*}(r) < 0 \ \ \forall r > 0. \tag{2.94}$$

It remains to prove that

$$\lim_{r \to 0^+} y_{k^*}(r) < \infty \text{ and } \lim_{r \to 0^+} y'_{k^*}(r) = 0. \tag{2.95}$$

Suppose, first of all, that $\lim_{r \to 0^+} y_{k^*}(r) = \infty$. Then $k^* \in S$, and it follows from the fact that S is open that $k^* \neq \sup S$, a contradiction. Thus it must be the case that $0 < \lim_{r \to 0^+} y_{k^*}(r) < \infty$. Finally, we assume for contradiction that $\lim_{r \to 0^+} y'_{k^*}(r) \neq 0$. Then either $L > 0$ and $r_3 > 0$ exist such that

$$y'_{k^*}(r) \leq -L \ \ \forall r \in (0, r_3), \tag{2.96}$$

or else

$$\liminf y'_{k^*}(r) < \limsup y'_{k^*}(r) = 0. \tag{2.97}$$

Assume that (2.96) holds, let $M = \lim_{r \to 0^+} (y_{k^*}(r))^p$, and let $r_3 > 0$ also satisfy

$$-\frac{L}{2} + \frac{M}{3} r_3 < 0. \tag{2.98}$$

It follows from (1.3) that $(r^2 y'_{k^*}(r))' \geq -r^2 M \ \ \forall r \in (0, r_3)$. An integration, combined with (2.98), gives $r^2 y'_{k^*}(r) \leq -\frac{L}{2} r_3^2 \ \ \forall r \in (0, r_3)$. A further integration gives

$$y_{k^*}(r) \geq y_{k^*}(r_3) + \frac{L}{2} r_3^2 \left(\frac{1}{r} - \frac{1}{r_3} \right) \ \ \forall r \in (0, r_3), \tag{2.99}$$

hence $\lim_{r\to 0^+} y_{k^*}(r) = \infty$, contradicting the fact that $y_{k^*}(r) \le M^{1/p} \ \ \forall r > 0$. Finally, suppose that (2.97) holds. Since $y_{k^*}(r) < M^{1/p}$ there is an $r_4 > 0$ such that

$$\frac{2}{p} - r^2 y_{k^*}^{p-1}(r) > 0 \ \ \forall r \in (0, r_4)\,. \tag{2.100}$$

Property (2.97) implies that $y''_{k^*}(r)$ changes sign infinitely often as $r \to 0^+$. Thus, there exists $r_5 \in (0, r_4)$ such that $y''_{k^*}(r_5) = 0$, and $y'''_{k^*}(r_5) \ge 0$, However, since $y'_{k^*}(r_5) < 0$, it follows from a differentiation of (1.3), and property (2.100) that $y'''_{k^*}(r_5) < 0$, a contradiction. This completes the proof of the lemma.

3 Conclusions

It is hoped that the "shooting from infinity" technique developed in this paper can be adapted to provide new insights into a wide variety of future research problems. For example, in **(I)-(III)** below we conjecture how Theorem 1.2 might be extended to a wider range of p and dimension N values. In Figure 2 (right panel) we describe how a natural "emergence" type phenomenon is observed as r decreases from $+\infty$.

(I) Extend the Theorem 1.2 to hold for all $N > 1$ and $2 < p < \frac{N+2}{N-2}$.

(II) Singular solutions and bound states with one or more zeros.

Let $N > 1$ and $2 < p < \frac{N+2}{N-2}$. Prove that a positive interval $(k_{N-1}, k_N]$ exists such that

(i) if $k \in (k_{N-1}, k_N)$ then $y_k(r)$ has exactly N positive zeros,

$$\lim_{r\to 0^+} y_k(r) = -\infty \ \text{ if N is odd}, \quad y_k(r) \to 0 \ \text{ as } \ r \to \infty \ \text{ if N is even}. \tag{3.1}$$

(ii) if $k = k_N$ then $y_{k_N}(r)$ is a bound state solution with exactly N positive zeros.

Figure 2 illustrates singular solutions and a bound state with one zero when $p = 3$.

(III) Uniqueness. A long standing unresolved problem is to prove uniqueness of bound states with more than one zero.

4 Appendix

The method of successive approximations is a standard technique (e.g. see Coddington and Levinson ([3], Chapter 13)) to prove existence and uniqueness of solu-

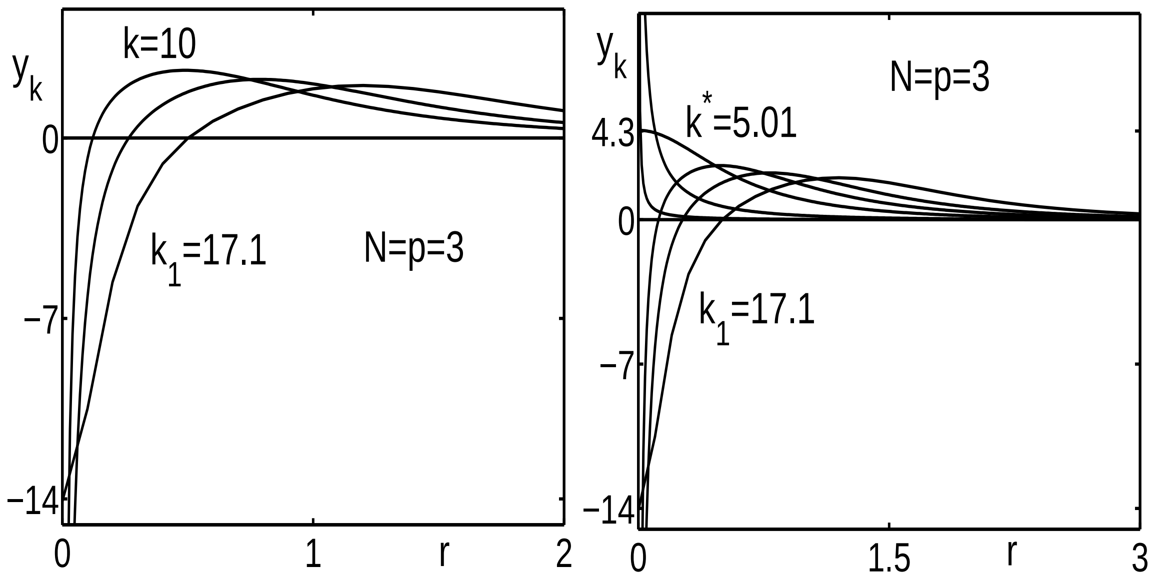

Figure 2: Solutions of integral equation (1.11) when $N = p = 3$; **(Left)** $k = 10, k = 14$ give singular solutions with one zero, $k_1 = 17.1$ gives a bound state with one zero. **(Right)** When $0 < k \leq k_1 = 17.1$ an emergence type phenomenon is observed; first, when $r \gg 1$ all solutions are small and behave linearly, i.e. $y_k \approx k\frac{e^{-r}}{r}$; second, as r decreases from large values nonlinear interactions cause solutions to ultimately begin growing in amplitude and emerge into four distinct types, **(1)** positive singular solutions when $0 < k < k^* \approx 5$, **(2)** the ground state when $k = k^*$, **(3)** singular solutions with one zero when $k^* < k < k_1 \approx 17.1$, **(4)** a bound state with one zero when $k = k_1$.

tions of integral equations. In the introduction we pointed out that Hastings and McLeod ([10], pp. 48-49) give complete details of the application of this method to prove existence of a unique solution of the integral equation associated with the Painleve type problem (1.12)-(1.13). The method of successive approximations applies equally well to prove Lemma 1.1, i.e. to prove, for each fixed $k > 0$ and $2 < p \leq 3$ that there exist $X_0 > 0$ and a unique solution of the integral equation

$$y_k(r) = k\frac{e^{-r}}{r} - \int_r^\infty \frac{t}{r}\sinh(t-r)y_k(t)|y_k(t)|^{p-1}dt, \quad r \geq X_0, \tag{4.1}$$

which satisfies

$$y_k(r) \sim k\frac{e^{-r}}{r} \quad \text{as} \quad r \to \infty, \tag{4.2}$$

and is also continuous in x and k. Although the proof is standard, our goal here is to give a brief sketch of the details of the application of the successive approximations method to prove existence. For this we follow precisely the same steps in the proof given by Hastings and McLeod ([10], pp. 48-49) for the Painleve problem.

Step 1. The first step is to derive two fundamental integral estimates, (4.6) and (4.7) below. Let $k > 0$ and $2 < p \leq 3$ be fixed, let $X > 0$, and write (4.1) as

$$y_k(r) = k\frac{e^{-r}}{r} - \frac{1}{2}\int_r^\infty \frac{t}{r}\left(e^{t-r} - e^{-(t-r)}\right)y_k(t)|y_k(t)|^{p-1}dt. \tag{4.3}$$

Next, define $y^0(r) = k\frac{e^{-r}}{r}$ $\forall r > 0$, and observe that

$$0 < te^t\left(y^0(t)\right)^p = \frac{k^p e^{-(p-1)t}}{t^{(p-1)}} \quad t \geq r \geq X. \tag{4.4}$$

Therefore $\int_r^\infty te^t\left(y^0(t)\right)^p dt$ converges for $r \geq X$, and

$$0 < \frac{e^{-r}}{2r}\int_r^\infty te^t\left(y^0(t)\right)^p dt = \frac{e^{-r}}{2r}\int_r^\infty \frac{k^p e^{-(p-1)t}}{t^{(p-1)}}dt \quad \forall r \geq X. \tag{4.5}$$

We conclude from (4.5) that $X_0 > 0$ can be chosen such that

$$0 < \frac{e^{-r}}{2r}\int_r^\infty te^t\left(y^0(t)\right)^p dt \leq \frac{k}{2^{p+1}}\frac{e^{-r}}{r}, \quad r \geq X_0. \tag{4.6}$$

Similarly, it follows from the fact that $e^{-2t} \leq e^{-2r}$ when $t \geq r > 0$, that

$$\begin{aligned} 0 < \frac{e^{r}}{2r}\int_r^\infty te^{-t}\left(y^0(t)\right)^p dt &= \frac{e^{-r}}{2r}\int_r^\infty \frac{k^p e^{2(r-t)}e^{-(p-1)t}}{t^{(p-1)}}dt \\ &\leq \frac{e^{-r}}{2r}\int_r^\infty \frac{k^p e^{-(p-1)t}}{t^{(p-1)}}dt \\ &\leq \frac{k}{2^{p+1}}\frac{e^{-r}}{r}, \quad r \geq X_0. \end{aligned} \tag{4.7}$$

Step 2. We now use (4.6) and (4.7) in our derivation of the appropriate uniformly bounded sequence of functions, (4.13) below. If

$$y^1(r) = k\frac{e^{-r}}{r} - \frac{1}{2}\int_r^\infty \frac{t}{r}\left(e^{t-r} - e^{-(t-r)}\right)\left(y^0(t)\right)^p dt, \tag{4.8}$$

then $y^1(r)$ is defined for $x \geq X_0$, and it follows from (4.6)-(4.7)-(4.8) that

$$|y^1(r)| \leq \left(k + \frac{2k}{2^{p+1}}\right)\frac{e^{-r}}{r} \leq 2k\frac{e^{-r}}{r},\ r \geq X_0. \tag{4.9}$$

Next, let $j \geq 1$ and suppose that $y^j(r)$ has been defined on $[X_0, \infty)$, and that

$$|y^j(r)| \leq 2k\frac{e^{-r}}{r},\ r \geq X_0. \tag{4.10}$$

Define

$$y^{j+1}(r) = k\frac{e^{-r}}{r} - \frac{1}{2}\int_r^\infty \frac{t}{r}\left(e^{t-r} - e^{-(t-r)}\right)\left(y^j(t)\right)^p dt,\ r \geq X_0. \tag{4.11}$$

It follows from (4.6), (4.7), (4.10) and (4.11) that

$$|y^{j+1}(r)| \leq 2k\frac{e^{-r}}{r},\ r \geq X_0. \tag{4.12}$$

We conclude from (4.8), (4.9), (4.10), (4.11) and (4.12) that

$$|y^j(r)| \leq 2k\frac{e^{-r}}{r} \quad \forall r \geq X_0 \ \text{ and } \ \forall j \geq 1. \tag{4.13}$$

Step 3. We use (4.13) in the completion of the proof. For each $j \geq 1$ define

$$y^{j+1}(r) - y^j(r) = -\int_r^\infty \frac{t}{2r}\left(e^{t-r} - e^{-(t-r)}\right)\left(\left(y^j(t)\right)^p - \left(y^{j-1}(t)\right)^p\right) dt, \tag{4.14}$$

where $r \geq X_0$. For each $j \geq 1$ and $t \geq r$ it follows from the mean value theorem that $\zeta^j(t)$ exists which satisfies

$$\left(y^j(t)\right)^p - \left(y^{j-1}(t)\right)^p = p\left(\zeta^j(t)\right)^{p-1}\left(y^j(t) - y^{j-1}(t)\right), \tag{4.15}$$

where

$$y^j(t) \leq \zeta^j(t) \leq y^{j+1}(t) \ \text{ if } \ y^j(t) \leq y^{j+1}(t), \tag{4.16}$$

$$y^{j+1}(t) \le \zeta^j(t) \le y^j(t) \ \text{ if } \ y^{j+1}(t) \le y^j(t). \tag{4.17}$$

Let $M = p2^{p-2}k^{p-1}$ and let $\|f\|$ denote the sup norm of f. From (4.13), (4.14), (4.15), (4.16), (4.17) and the fact that $p > 2$, we conclude that X_0 can be further chosen such that if $r \ge X_0$ then

$$\begin{aligned}
|y^{j+1}(r) - y^j(r)| &\le \int_r^\infty \frac{pt}{2r}\left(e^{t-r} - e^{-(t-r)}\right)|\zeta^j(t)|^{p-1}|y^j(t) - y^{j-1}|dt \\
&\le \int_r^\infty \frac{Mt}{r}\left(e^{t-r} - e^{-(t-r)}\right)\frac{e^{-(p-1)t}}{t^{p-1}}|y^j(t) - y^{j-1}(t)|dt \\
&\le \int_r^\infty \frac{M}{r}e^{t-r}\frac{e^{-(p-1)t}}{t^{p-2}}|y^j(t) - y^{j-1}(t)|dt \\
&\le \left\|y^j - y^{j-1}\right\| \int_r^\infty \frac{M}{r}e^{t-r}\frac{e^{-(p-1)t}}{t^{p-2}}dt \\
&\le \left\|y^j - y^{j-1}\right\| \int_r^\infty \frac{Me^{-r}}{r}\frac{e^{-(p-2)t}}{t^{p-2}}dt < \frac{1}{2}\left\|y^j - y^{j-1}\right\|.
\end{aligned} \tag{4.18}$$

Thus,

$$\left\|y^{j+1} - y^j\right\| \le \frac{1}{2}\left\|y^j - y^{j-1}\right\| \quad \forall j \ge 1, \tag{4.19}$$

and we conclude that $y^0 + \sum_{j=0}^{\infty}(y^{j+1} - y^j)$ converges uniformy on $[X_0, \infty)$ to a continuous function $y_k(r)$ which solves (4.3). As pointed out by Hastings and McLeod ([10], pp. 48-49), "uniqueness and continuity with respect to k follow in the usual way for proofs using successive approximations (Coddingtion [3], Chapter 13)."

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