Radially Symmetric Solutions of $\Delta w - |w|^{p-1}w = 0$

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Abstract

We investigate the existence and asymptotic behavior of radially symmetric singular solutions of $w'' + \frac{N-1}{r}w' - |w|^{p-1}w = 0$, $r > 0$, which are unbounded at $r = 0$. We focus on the parameter regime $N > 2$ and $1 < p < \frac{N}{N-2}$ where the equation has the singular solution $w_1 = \left(\frac{4 - 2(N-2)(p-1)}{(p-1)^2}\right)^{\frac{1}{p-1}} \frac{1}{r^{\frac{N}{N-2}}}$, $r > 0$.

Given the above stated parameter regime, we prove the existence and analyze the asymptotic behavior of new positive singular solutions.

Keywords: radially symmetric, singular solution

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1. Introduction

We investigate the behavior of solutions of

$$\Delta w - |w|^{p-1}w = 0,$$  \hspace{1cm} (1.1)

where $w = w(x_1, \ldots, x_N)$, $N > 1$ and $p > 1$. Solutions of (1.1) are time independent solutions of the nonlinear heat equation

$$\frac{\partial w}{\partial t} = \Delta w - |w|^{p-1}w.$$  \hspace{1cm} (1.2)

In the mid 1980’s Brezis, Peletier and Terman [1], and Kamin and Peletier [7] investigated the existence and asymptotic behavior of a special class of posi-
tive solutions of (1.2) which are known as 'very singular solutions.' These solutions exist in the parameter regime $N > 2$ and $1 < p < \frac{N+2}{N}$. Subsequently, Kamin, Peletier and Vazquez [8] extended these results and studied (1.2) with the stated goal of classifying all nonnegative singular solutions.

In this paper we extend the results in [8], and investigate the existence and asymptotic behavior of time independent, radially symmetric solutions of (1.1). Such solutions have the form $w = w(r)$, where $r = (x_1^2 + \cdots + x_N^2)^{1/2}$, and satisfy

$$w'' + \frac{N-1}{r}w' - |w|^{p-1}w = 0, \quad r > 0. \quad (1.3)$$

Following [8], we focus on positive singular solutions of (1.3). We mention three examples. First, consider solutions which are bounded at $r = 0$, and satisfy $(w(0), w'(0)) = (w_0, 0)$, where $w_0$ is finite. These solutions become singular, i.e. they blow up, at finite, positive $r$ values. A second class consists of solutions of (1.3) that are unbounded at $r = 0$. A third type of positive singular solution is one which becomes unbounded at two positive values of $r$.

Equation (1.3) has the well known positive singular solution

$$w_1(r) = \left(\frac{4 - 2(N-2)(p-1)}{(p-1)^2}\right)^{\frac{1}{p-1}} r^{\frac{2-p}{p-1}}, \quad N > 2, \quad 1 < p < \frac{N}{N-2}. \quad (1.4)$$

Note that, when $N > 2$, this solution exists $\forall p \in (1, \frac{N}{N-2})$, whereas the 'very singular' solutions in [1] exist on the smaller interval $1 < p < \frac{N+2}{N}$.

The function $w_1(r)$ has the following 'local integrability' property: for each $L > 0$,

$$0 < \int_0^L r^{N-1}w_1^q(r)dr < \infty \iff 0 < q < \frac{N(p-1)}{2}. \quad (1.5)$$

For example, when $N = 3$ and $q = 1$,

$$0 < \int_0^L r^2w_1(r)dr < \infty \text{ when } \frac{5}{3} < p < 3. \quad (1.6)$$

It is interesting to observe that, when $N = 3$ and $q = 1$, the 'very singular' solutions exist on the interval $1 < p < \frac{5}{3}$, which is outside the range of integrability given in (1.6).
Finally, we find that $w_1(r)$ is not globally integrable for any $q > 0$. That is,

$$\int_0^{\infty} r^{N-1} w_1^q(r) \, dr = \infty \quad \forall \, q > 0, \; N > 2, \; 1 < p < \frac{N}{N-2}. \quad (1.7)$$

**A related equation.** A second, widely studied nonlinear heat equation is

$$\frac{\partial v}{\partial t} = \Delta v + |v|^{p-1}v. \quad (1.8)$$

This equation also has a radially symmetric solution which is positive and singular at $r = 0$, namely

$$v_1(r) = \left(\frac{2(N-2)(p-1) - 4}{(p-1)^2}\right)^{\frac{1}{p-1}} r^\frac{2}{p}, \; N > 2, \; \frac{N}{N-2} < p < \frac{N+2}{N-2}. \quad (1.9)$$

This well known solution has played an important role in the analysis of blowup of solutions of (1.8). For example, when $v(x_1, \ldots, x_N, 0)$ is appropriately chosen, similarity solutions methods show how $v(x_1, \ldots, x_N, t) \to cv_1(r)$ as $t \to \infty$, where $c > 0$ is a constant [4, 5, 10]. In a separate paper [11] we proved the existence and asymptotic behavior of positive singular, radially symmetric solutions other than $v_1(r)$. These may also play an important role in the analysis of (1.8).

**Specific Aims.** We assume throughout that $N > 2$ and $1 < p < \frac{N}{N-2}$, and investigate the following fundamental issues:

Do positive singular solutions exist which are different from $w_1(r)$? What is their asymptotic behavior as $r \to 0^+$, and as $r \to \infty$? How are their integrability properties similar to, or different from, those of $w_1(r)$?

**Methods of Analysis.** One approach to answering the questions posed above is to let $r_0 > 0$ be arbitrarily chosen and fixed, and assume that

$$w(r_0) = a > 0 \quad \text{and} \quad w'(r_0) = b \in \mathbb{R}. \quad (1.10)$$

Our goal is to determine the behavior of solutions of (1.3)-(1.10) over the largest interval $(r_{\min}, r_{\max})$ containing $r_0$ where the solution is positive. Thus,

$$r_{\min} = \inf\{r \in (0, r_0) | w(r) > 0 \; \forall r \in ]r, r_0]\}, \quad (1.11)$$
and
\[ r_{\text{max}} = \max \{ \hat{r} > r_0 | w(r) > 0 \ \forall r \in ([r_0, \hat{r}]) \}. \quad (1.12) \]

**Our Approach.** Because (1.3) is non-autonomous, it is difficult to use methods such as topological shooting, or Pohozaev identity arguments [2, 9], to prove the existence of new singular solutions. Our approach to investigating the questions raised in (I) - (II) is to derive a related equation which is *autonomous* and more amenable to analysis. For this, let \( w(r) \) denote any solution of (1.3), and define
\[ h(\tau) = \frac{w(\exp(\tau))}{w_1(\exp(\tau))}, \quad -\infty < \tau < \infty. \quad (1.13) \]

Then \( h(\tau) \) solves
\[ \frac{d^2 h}{d\tau^2} + \frac{N - 2}{p - 1} \left( p - \frac{N + 2}{N - 2} \right) \frac{dh}{d\tau} + \frac{2(N - 2)}{(p - 1)^2} \left( p - \frac{N}{N - 2} \right) (|h|^{p-1} - 1)h = 0. \quad (1.14) \]

Because (1.14) is autonomous, we can apply phase plane techniques to determine the behavior of solutions. We then use the ‘inverse’ formula
\[ w(r) = w_1(r)h(\ln(r)), \quad 0 < r < \infty \quad (1.15) \]

to obtain corresponding solutions of the \( w \) equation (1.3). In particular, we will prove that there is a heteroclinic orbit leading from \((0, 0)\) to \((1, 0)\) in the \((h, h')\) phase plane (see Figure 2). We then use (1.15) to show that there is a value \( D > 0 \) such that the corresponding solution, \( w_2(r) \), of (1.3) is a positive singular solution, with \( w_2(r) < w_1(r) \ \forall r > 0 \), and
\[
\frac{w_2(r)}{w_1(r)} \to 1 \text{ as } r \to \infty \quad \text{and} \quad \frac{w_2(r)}{w_1(r)} \sim Dr^{\frac{N-2}{p-1}}(\frac{N}{N-2}-p) \to 0 \text{ as } r \to 0. \quad (1.16)
\]

Thus, \( w_2(r) \sim w_1(r) \) as \( r \to \infty \), and \( w_1(r) \to \infty \) faster than \( w_2(r) \) as \( r \to 0^+ \).

2. **The Main Result**

In this section we analyze solutions of the \( h \) equation (1.14), and the corresponding solutions of the \( w \) equation (1.3) when \( N > 2 \) and \( 1 < p < \frac{N}{N-2} \).

Our goals:
I. In Theorem 2.1, we classify the behavior of solutions of (1.14). We focus on solutions whose trajectories in the \((h, \frac{dh}{d\tau})\) phase plane lead from the asymptotically unstable fixed point \((0, 0)\) to either one of the two saddle points \((\pm 1, 0)\) (see upper left panel of Figure 2). These solutions translate into new singular solutions of (1.3) by way of (1.13). In particular, the solution of (1.14) leading from the constant solution \(h \equiv 0\) to the constant solution \(h \equiv 1\) translates into a new positive singular solution, denoted by \(w_2(r)\), of (1.3) (see Figure 2, second row, right panel).

II. We determine the asymptotic behavior of \(w_2(r)\) as \(r \to 0^+\) and as \(r \to \infty\).

III. We obtain parameter regimes that ensure that \(w_2(r)\) satisfies local and global integrability properties.

Figure 2 illustrates the behavior of solutions when \((N, p) = (3, 2)\).

**Theorem 2.1.** Let \(N > 2\) and \(1 < p < \frac{N}{N - 2}\). Then

(i) There is a two dimensional unstable manifold of solutions of (1.14) that lead from the constant solution \((0, 0)\) into the \((h, \frac{dh}{d\tau})\) phase plane.

(ii) There is a one dimensional stable manifold of solutions leading to \((1, 0)\) in the \((h, \frac{dh}{d\tau})\) phase plane. One component, \(B_1\), points into the \(h < 1, \frac{dh}{d\tau} > 0\) region of the phase plane, and its negative counterpart, \(B_2\), points into the \(h > -1, \frac{dh}{d\tau} < 0\) region (see Figure 2, upper left). If \((h(0), h'(0)) \in B_1\), then

\[
0 < h(\tau) < 1 \quad \text{and} \quad 0 < h'(\tau) < \frac{N - 2}{p - 1} \left( \frac{N}{N - 2} - p \right) h(\tau) \quad \forall \tau \in \mathbb{R}, \quad (2.1)
\]

\[
\lim_{\tau \to -\infty} (h(\tau), h'(\tau)) = (0, 0) \quad \text{and} \quad \lim_{\tau \to -\infty} \frac{h'(\tau)}{h(\tau)} = \frac{N - 2}{p - 1} \left( \frac{N}{N - 2} - p \right), \quad (2.2)
\]

\[
\lim_{\tau \to -\infty} (h(\tau), h'(\tau)) = (1, 0). \quad (2.3)
\]

Proof. (i) A linearization of (1.14) around the constant solution \(h \equiv 0\) gives

\[
\frac{d^2h}{d\tau^2} + \frac{N - 2}{p - 1} \left( p - \frac{N + 2}{N - 2} \right) \frac{dh}{d\tau} = 2 \frac{(N - 2)}{(p - 1)^2} \left( p - \frac{N}{N - 2} \right) h = 0. \quad (2.4)
\]

The eigenvalues associated with (2.4) satisfy

\[
\mu_1 = \frac{N - 2}{p - 1} \left( \frac{N}{N - 2} - p \right) > 0 \quad \text{and} \quad \mu_2 = \frac{2}{p - 1} > 0. \quad (2.5)
\]
The existence of the two dimensional unstable manifold of solutions is an immediate consequence of the Stable Manifold Theorem [3]. We will make use of the observation that (1.14) can be written as

\[ \frac{d^2 h}{d\tau^2} - (\mu_1 + \mu_2) \frac{dh}{d\tau} + \mu_1 \mu_2 h = \mu_1 \mu_2 |h|^{p-1} h. \]  

(2.6)

(ii) Linearizing (1.14) about the positive constant solution \( h \equiv 1 \) gives

\[ \frac{d^2 h}{d\tau^2} + \frac{N - 2}{p - 1} \left( p - \frac{N + 2}{N - 2} \right) \frac{dh}{d\tau} + \frac{2(N - 2)}{p - 1} \left( p - \frac{N}{N - 2} \right) (h - 1) = 0. \]

(2.7)

Define \( k = -\frac{2}{p - 1} \). Then (2.7) becomes

\[ \frac{d^2 h}{d\tau^2} + \gamma \frac{dh}{d\tau} + 2(\gamma - k)(h - 1) = 0, \]

(2.8)

where

\[ \gamma = \frac{N - 2}{p - 1} \left( p - \frac{N + 2}{N - 2} \right) < 0 \quad \text{and} \quad \gamma - k = \frac{N - 2}{p - 1} \left( p - \frac{N}{N - 2} \right) < 0. \]

(2.9)

Thus, the eigenvalues associated with (2.7) and (2.8) satisfy

\[ \lambda_1 = -\gamma - \sqrt{\gamma^2 - 8(\gamma - k)} < 0 \quad \text{and} \quad \lambda_2 = -\gamma + \sqrt{\gamma^2 - 8(\gamma - k)} > 0. \]

(2.10)

It follows from (2.10) and the Stable Manifold Theorem that there is a one dimensional stable manifold of solutions leading to (1,0) in the \((h, \frac{dh}{d\tau})\) phase plane. Additionally,

\[ \lim_{\tau \to \infty} \frac{h'(\tau)}{h(\tau) - 1} = \lambda_1 \]

(2.11)

for \((h(\tau), h'(\tau))\) on the stable manifold. Thus, for sufficiently large \( \tau \), solutions on the stable manifold leading to (1,0) satisfy \( h(\tau) > 1 \) if \( h'(\tau) < 0 \) and \( h(\tau) < 1 \) if \( h'(\tau) > 0 \). Let \( B_1 \) denote the component of the stable manifold with \( h(\tau) < 1 \) and \( h'(\tau) > 0 \) (see top row of Figure 2). Assume throughout that \((h(0), h'(0)) \in B_1\), and hence (2.3) holds.

It remains to show that (2.1) and (2.2) hold. Because of (2.11) and the translation invariance of (2.6), we can choose \( 1 - h(0) > 0 \) and \( h'(0) > 0 \) sufficiently small so that

\[ 0 < h(\tau) < 1 \quad \text{and} \quad 0 < h'(\tau) < \mu_1 h(\tau) \quad \text{on} \quad [0, \infty). \]

(2.12)
The definition of $B_1$, together with (2.12), imply that the maximal interval of existence is of the form $(\tau_{\text{min}}, \infty)$ where $\tau_{\text{min}} < 0$.

Next, we show that $B_1 \subset U^o$ where $U^o$ is the bounded open triangular region

$$U^o = \{(h_1, h_2) \mid 0 < h_1 < 1 \text{ and } 0 < h_2 < \mu_1 h_1\}. \quad (2.13)$$

Figure 2 (upper right) shows $U^o$ when $N = 3$ and $p = 2$. Because of (2.12), it is sufficient to show that $(h(\tau), h'(\tau)) \in U^o$ $\forall \tau \in (\tau_{\text{min}}, 0]$. For a contradiction, assume that $(h(\tau), h'(\tau))$ leaves $U^o$ at some point in $(\tau_{\text{min}}, 0)$. Define the functional

$$H = \frac{dh}{d\tau} - \mu_1 h. \quad (2.14)$$

It follows from (2.6) that $H$ satisfies

$$H' - \mu_2 H = \mu_1 \mu_2 |h|^{p-1} h. \quad (2.15)$$

Suppose that $(h(\tau), h'(\tau))$ leaves $U^o$ across the line $H = 0$. That is, (see Figure 1, left panel) suppose that there exists $\tau_0 \in (\tau_{\text{min}}, 0)$ such that

$$H(\tau) < 0 \text{ and } 0 < h(\tau) < 1 \text{ on } (\tau_0, 0), \text{ and } H(\tau_0) = 0. \quad (2.16)$$

If $h(\tau_0) = 0$, then (2.14) implies that $h'(\tau_0) = 0$, contradicting uniqueness of the constant solution $(h(\tau), h'(\tau)) \equiv (0, 0)$. Thus, $h(\tau_0) > 0$. Also, (2.16) implies that

$$H'(\tau_0) \leq 0. \quad (2.17)$$
The fact that \( h(\tau_0) > 0 \), combined with (2.15), results in
\[
H'(\tau_0) = \mu_1\mu_2(h(\tau_0))^p > 0,
\]
a contradiction of (2.17). Thus, \((h(\tau), h'(\tau))\) can only leave \( U^o \) across the line segment \( 0 < h < 1 \), \( h' = 0 \). If so, there is a value \( \tau_1 \in (\tau_{min}, 0) \) such that
\[
H(\tau) < 0, \ 0 < h(\tau) < 1, \ \text{and} \ h'(\tau) > 0 \ \forall \ \tau \in (\tau_1, 0), \quad (2.18)
\]
and
\[
0 < h(\tau_1) < 1 \ \text{and} \ h'(\tau_1) = 0, \quad (2.19)
\]
as depicted in the right panel of Figure 1. Hence,
\[
h''(\tau_1) \geq 0. \quad (2.20)
\]
It follows from (2.6) and (2.19) that
\[
h''(\tau_1) = \mu_1\mu_2((h(\tau_1))^{p-1} - 1)h(\tau_1) < 0,
\]
contradicting (2.20). We conclude that \((h(\tau), h'(\tau))\) cannot leave \( U^o \) on \((\tau_{min}, \infty)\), hence \( B_1 \subset U^o \) as claimed. Moreover, since \((h(\tau), h'(\tau))\) is bounded, then \( \tau_{min} = -\infty \) follows from standard ODE theory. Thus, \((h(\tau), h'(\tau))\) \( \in U^o \ \forall \ \tau \in \mathbb{R} \), and therefore \( h'(\tau) > 0 \ \forall \ \tau \in \mathbb{R} \). This completes the proof of (2.1).

**Proof of (2.2).** First, we prove that \( h \to 0^+ \) as \( \tau \to -\infty \). Since \( h'(\tau) > 0 \) and \( 0 < h(\tau) < 1 \) on \( \mathbb{R} \), then \( 0 \leq \bar{h} < 1 \) where \( \bar{h} = \lim_{\tau \to -\infty} h \). To obtain a contradiction suppose that \( \bar{h} > 0 \). Then \( 0 < \bar{h} < 1 \) and (2.6) yield
\[
\frac{d^2h}{d\tau^2} - (\mu_1 + \mu_2)\frac{dh}{d\tau} \to \mu_1\mu_2(\bar{h}^{p-1} - 1)\bar{h} < 0 \ \text{as} \ \tau \to -\infty. \quad (2.21)
\]
It follows from (2.21) that \( h'(\tau) - (\mu_1 + \mu_2)h(\tau) \to \infty \) as \( \tau \to -\infty \) which contradicts the fact that \( U^o \) is bounded and \((h(\tau), h'(\tau)) \) \( \in U^o \) for all \( \tau \in \mathbb{R} \). Thus, \( h(\tau) \to 0^+ \) as \( \tau \to -\infty \). Next, we show that \( h'(\tau) \to 0^+ \) as \( \tau \to -\infty \). Note that \( 0 < h'(\tau) < \mu_1h(\tau) \) on \((-\infty, 0]\) is an immediate consequence of \( H(\tau) < 0 \) and \( h'(\tau) > 0 \) on \((-\infty, 0] \). Therefore, \( h'(\tau) \to 0^+ \) as \( \tau \to -\infty \) follows from the fact that \( h(\tau) \to 0^+ \) as \( \tau \to -\infty \).

**Proof of (2.3).** Finally, we need to prove that \( \rho = \frac{h'}{h} \to \mu_1 \) as \( \tau \to -\infty \). The definition of \( \rho \) together with (2.6) gives
\[
\rho' + \rho^2 - (\mu_1 + \mu_2)\rho = \mu_1\mu_2(h^{p-1} - 1). \quad (2.22)
\]
We now show that $\rho \to \mu_1$ monotonically as $\tau \to -\infty$. Differentiating (2.22) yields
\[
\rho'' + (2\rho - \mu_1 - \mu_2)\rho' = \mu_1\mu_2(p - 1)h^{p-2}h'.
\]
Hence, if $\rho'(\tau_s) = 0$ for some $\tau_s \in \mathbb{R}$, then
\[
\rho''(\tau_s) = \mu_1\mu_2(p - 1)h^{p-2}(\tau_s)h'(\tau_s) > 0.
\]
This implies that $\rho'$ has at most one zero on $\mathbb{R}$. Furthermore,
\[
0 < \rho(\tau) = \frac{h'(\tau)}{h(\tau)} < \mu_1 \quad \text{for all} \quad \tau \in \mathbb{R}
\]
since
\[
(h(\tau), h'(\tau)) \in U^o \quad \text{for all} \quad \tau \in \mathbb{R}.
\]
Thus, $\bar{\rho} = \lim_{\tau \to -\infty} \rho$ exists and $0 \leq \bar{\rho} \leq \mu_1$. Moreover, the fact that $\bar{\rho}$ is finite ensures the existence of an unbounded decreasing sequence $\{\tau_n\}$ such that $\lim_{\tau_n \to -\infty} \rho'(\tau_n) = 0$. Substituting
\[
\lim_{\tau_n \to -\infty} \rho'(\tau_n) = 0 = \lim_{\tau_n \to -\infty} h(\tau_n) \quad \text{and} \quad \bar{\rho} = \lim_{\tau_n \to -\infty} \rho
\]
into (2.22) results in
\[
\bar{\rho}^2 - (\mu_1 + \mu_2)\bar{\rho} + \mu_1\mu_2 = 0. \tag{2.23}
\]
The bound $0 \leq \bar{\rho} \leq \mu_1$ and (2.23) imply that $\bar{\rho} = \mu_1$. Thus, $\rho \to \mu_1$ as $\tau \to -\infty$ as claimed. This completes the proof of the theorem.

**Implications for the $w$ equation.** Part (ii) of Theorem 2.1 has the following important implications for solutions of (1.3):

1. **A new positive singular solution.** Let $h_2$ denote a solution of (1.3) which satisfies part (ii) of Theorem 2.1. By (1.15), the solution of (1.3) corresponding to $h_2$ is
\[
w_2(r) = w_1(r)h_2(\ln(r)). \tag{2.24}
\]
Since $0 < h_2(\ln(r)) < 1 \quad \forall r > 0$, it follows from (2.24) that
\[
0 < w_2(r) < w_1(r) \quad \forall r > 0 \tag{2.25}
\]
(see Figure 2, right panel, second row).
Figure 2: \( N=3, p=2 \). First row: trajectories of the unstable manifold \((A_1 \text{ and } A_2)\) and stable manifold \((B_1 \text{ and } B_2)\) leading from \((0,0)\) in the \((h, \frac{dh}{d\tau})\) phase plane. Second and third rows: the \(h\) components of solutions along \(A_1, A_2, B_1, B_2\) (left column) and corresponding \(w\) components along \(A_1\) and \(B_1\) (right column): \(w_0(r)\) is bounded at \(r = 0\), \(w_1(r) = 2r^{-2}\) is the known singular solution, and \(w_2(r)\) denotes the new, positive singular solution corresponding to the heteroclinic orbit solution labeled \(B_1\).
We claim that $w_2$ is singular at $r = 0$. The first step in proving this claim is to observe that properties (2.2) and (2.5) imply that $\frac{h_2(\tau)}{h_2(\tau)} \sim \mu_1$ as $\tau \to -\infty$. Therefore, $\ln(h_2(\tau)) \sim \mu_1 \tau$ as $\tau \to -\infty$. This together with the fact that $\tau = \ln(r)$ lead to

$$h_2(\tau) = h_2(\ln(r)) \sim r^{\mu_1} \text{ as } r \to 0^+.$$  

(2.26)

Combining (1.4) with (2.5) gives $w_1(r) = \left(\frac{4 - 2(N - 2)(p - 1)}{(p - 1)^2}\right)^{\frac{1}{p-1}} r^{-u_2}$. Thus,

$$w_2(r) \sim \left(\frac{4 - 2(N - 2)(p - 1)}{(p - 1)^2}\right)^{\frac{1}{p-1}} r^{\mu_1 - \mu_2} \text{ as } r \to 0^+$$  

(2.27)

is a consequence of (2.24) and (2.26). Our claim that $w_2$ is singular at $r = 0$ follows from (2.27) and the fact that $\mu_1 - \mu_2 = 2 - N < 0$.

Next, we determine the asymptotic behavior of $w_2(r)$ as $r \to \infty$. Since $h_2(\ln(r)) \to 1^-$ as $r \to \infty$ then $\frac{w_2(r)}{w_1(r)} \sim 1$ as $r \to \infty$. Therefore,

$$w_2(r) \sim \left(\frac{4 - 2(N - 2)(p - 1)}{(p - 1)^2}\right)^{\frac{1}{p-1}} r^{-u_2} \text{ as } r \to \infty.$$  

(2.28)

(2) Integral properties of $w_2(r)$. First, we determine a parameter regime that gives rise to the local integrability property

$$0 < \int_0^L r^{-N-1} w_2^q(r) dr < \infty,$$

where $L > 0$ is fixed and arbitrary and $q > 0$. It follows immediately from (2.7) that

$$\int_0^L r^{-N-1} w_2^q(r) dr < \infty \iff N - 1 + q(\mu_1 - \mu_2) > -1.$$  

(2.29)

Since $\mu_1 - \mu_2 = 2 - N < 0$, then (2.29) is equivalent to

$$\int_0^L r^{-N-1} w_2^q(r) dr < \infty \iff q < \frac{N}{\mu_2 - \mu_1}.$$  

(2.30)

Finally, we determine a range of parameters that ensures the global integrability property

$$0 < \int_0^\infty r^{-N-1} w_2^q(r) dr < \infty.$$  

(2.31)
where $q > 0$. To accomplish this, express the integral in (2.31) as
\[ \int_0^\infty r^{N-1}w_2^q(r)dr = \int_0^L r^{N-1}w_2^q(r)dr + \int_L^\infty r^{N-1}w_2^q(r)dr, \] (2.32)
where $L > 0$ can be chosen arbitrarily. It follows from (2.28) that
\[ \int_L^\infty r^{N-1}w_2^q(r)dr < \infty \iff N - 1 - q\mu_2 < -1, \]
or equivalently
\[ \int_L^\infty r^{N-1}w_2^q(r)dr < \infty \iff \frac{N}{\mu_2} < q. \] (2.33)
Combining (2.30), (2.32), and (2.33) results in
\[ \int_0^\infty r^{N-1}w_2^q(r)dr < \infty \iff \frac{N}{\mu_2} < q < \frac{N}{\mu_2 - \mu_1}. \] (2.34)

Alternatively, we can use the fact that $\mu_2 = \frac{2}{p-1}$ and $\mu_2 - \mu_1 = N - 2$ to express the bounds on $q$ given in (2.30) and (2.34)
\[ \int_0^L r^{N-1}w_2^q(r)dr < \infty \iff q < \frac{N}{N - 2}, \]
and
\[ \int_0^\infty r^{N-1}w_2^q(r)dr < \infty \iff \frac{N(p - 1)}{2} < q < \frac{N}{N - 2}. \]

3. Conclusion

In this paper we proved the existence of a new positive singular solution, $w_2$. Our analytic approach consisted of the following four steps:

**Step 1** We transformed the non-autonomous equation (1.3) to the autonomous equation (1.14).

**Step 2** We applied phase plane techniques to prove that (1.14) has a positive increasing heteroclinic orbit solution leading from $(h, h') = (0, 0)$ to $(h, h') = (1, 0)$ in the $(h, h')$ phase plane.

**Step 3** We showed that the solution $w_2$ of (1.3) associated with the heteroclinic orbit solution of (1.14) is positive and singular at $r = 0$. 


Step 4 Finally, we determined parameter regimes that ensure that our new positive singular solution satisfies local and global integrability properties. The parameter regimes that ensure local and global integrability of $w_2$ are given in (2.30) and (2.34) respectively.

It would be interesting to determine the role that $w_2(r)$ might play in the analysis of the time dependent PDE (1.2). For example, can the analytic techniques developed by Galaktionov and Vazquez [4], and Souplet and Weissler [10] be extended to apply to $w_2(r)$?
References


$\Delta w + |w|^{p-1}w = 0$ preprint, 2010